11/20

- Teacher evaluation:

  Please do it and send me an e-mail when you submit it.

- Final Exam:

  - Come to class on Tuesday 12/4 to pick it up.
  - Turn it in in our classroom at the time of the final exam: Thursday 12/6 at 2:45 PM!

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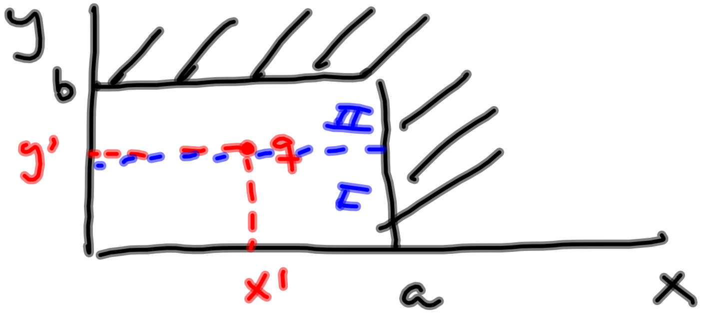
Green Functions.

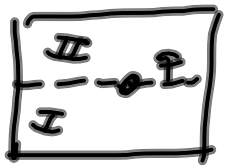


Find $G_D(x, x', y, y')$

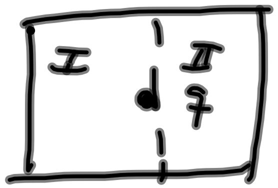
$\phi = 0$ on all surfaces.

$q = 4\pi\varepsilon_0$

Solve the problem for the potential of q inside V using separation of variables



or



both are equivalent exchanging $x \leftrightarrow y$, $a \leftrightarrow b$.

Propose:

$$\phi^{I}(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

$$\phi^{II}(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \sinh\left[(b-y)\frac{n\pi}{a}\right]$$

At $y = y'$ $\phi^{I} = \phi^{II}$

$$\phi^{I}(x, y=y') = \phi^{II}(x, y=y')$$

and

$$-\frac{\partial\phi^{II}}{\partial n} + \frac{\partial\phi^{I}}{\partial n} = \frac{\rho(\bar{x})}{\varepsilon_0}$$

$$\rho(\bar{x}) = q\,\delta(x-x')$$
$$= 4\pi\varepsilon_0\,\delta(x-x')$$

$n = y$

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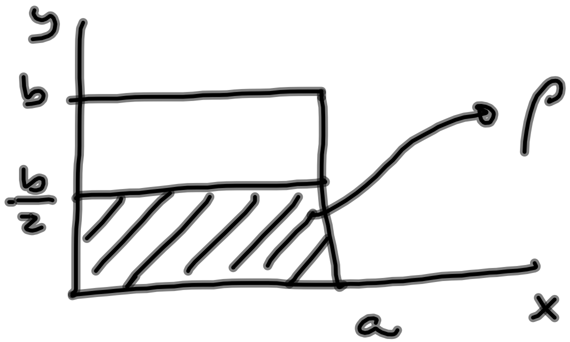
$$-\left.\frac{\partial \phi^{II}}{\partial y}\right|_{y=y'} + \left.\frac{\partial \phi^{I}}{\partial y}\right|_{y=y'} = 4\pi\, \delta(x-x')$$

In this expression use orthogonality multiplying both sides by $\sin\frac{n\pi x}{a}$ and integrating over x. You'll obtain:

$$G(x,x',y,y') = 8 \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \frac{\sinh\left[(b-y_>)\frac{n\pi}{a}\right] \sinh\frac{n\pi y_<}{a}}{\sinh\frac{bn\pi}{a}}$$

$y_<$ ($y_>$) is the smaller (larger) of y and y'.

Homewor problem #3:



$$\rho(\vec{x}) = \sigma_0\, \underbrace{\Theta\left(y-\frac{b}{2}\right)}_{\begin{cases} 1 & \text{if } y \le \frac{b}{2} \\ 0 & \text{if } y > b/2 \end{cases}}$$

Find $\phi(x,y)$ in V if $\phi|_S = 0$ in all surfaces.

$$\phi(x,y) = \frac{1}{4\pi\varepsilon_0}\int_0^a dx' \int_0^b dy'\, G(x,x',y,y')\,\rho(x',y') =$$

$$= \frac{\sigma_0}{4\pi\varepsilon_0}\int_0^a dx' \int_0^{b/2} dy'\, G(x,x',y,y')$$

• For $y > \frac{b}{2}$ use G with $y_> = y$ and $y_< = y'$.

• For $y < \frac{b}{2}$ more work is needed:

$$\phi(x,y) = \frac{\sigma}{4\pi\varepsilon_0} \int_0^a dx' \Bigg[\int_0^y dy' \, \underbrace{G(x,x',y,y')}_{\substack{\text{with } y_< = y' \\ y_> = y}} + \int_y^{b/2} dy' \, \underbrace{G(x,x',y,y')}_{\substack{\text{with } y_< = y \\ y_> = y'}} \Bigg]$$

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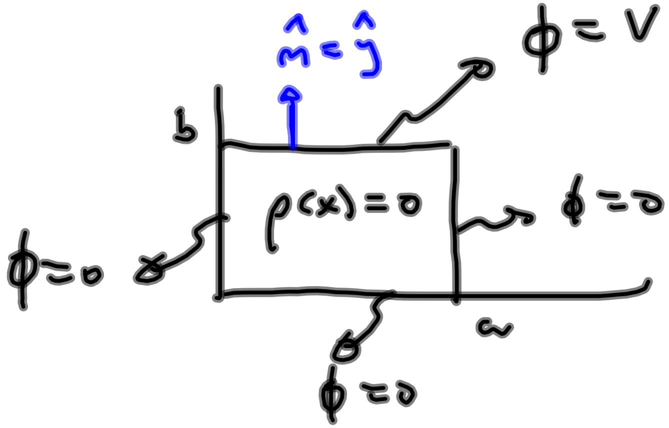
Another problem:



$$\phi(x,y) = \frac{1}{4\pi\varepsilon_0} \int_V G(\bar{x}, \bar{x}') \overbrace{\rho(\bar{x}')}^{0} \, d^3x' -$$

$$- \frac{1}{4\pi} \oint ds' \, \phi_s \frac{\partial G}{\partial n'} \Big|_s \qquad \hat{n}' = y'$$

$$\Phi(x,y) = -\frac{V}{4\pi}\int_0^a \frac{\partial G}{\partial y'}\bigg|_{y'=b} dx'$$

use G with $y_< = y$ $\quad y_> = y'$

since $y' = b$.

$$G(\bar{x},\bar{y}) = 8\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}$$

$$\frac{\sinh\left[(b-y')\frac{n\pi}{a}\right]\sinh\frac{n\pi y}{a}}{\sinh\frac{bn\pi}{a}}$$

$$\frac{\partial G}{\partial y'}\bigg|_{y'=b}$$

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You should obtain:

$$\phi(x,y) = \frac{4V}{\pi} \sum_{j=0}^{\infty} \frac{\sin \frac{(2j+1)\pi x}{a} \sinh \frac{(2j+1)\pi y}{a}}{(2j+1) \sinh \frac{b(2j+1)\pi}{a}}$$

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Integral Transformations Ch. 15.

$$\underbrace{g(\alpha)}_{\text{integral transform}} = \int_a^b \underbrace{f(t)}_{\text{function}} \underbrace{K(\alpha, t)}_{\text{kernel}} \, dt$$

This transformation provides a mapping of a function $f(t)$ into another function $g(\alpha)$.

α and t are called conjugate variables.

Examples: $(\alpha, t) = (t, \omega)$ or (x, k)

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Fourier Transform:

Kernel: $K(\alpha, t) = e^{-i\alpha t}/\sqrt{2\pi}$

or $K(\omega, t) = \dfrac{e^{-i\omega t}}{\sqrt{2\pi}}$

then

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \qquad (1)$$

$f(t)$ is normally a wave packet concentrate around t_0 and $g(\omega)$ provides its structure in terms of plane waves of frequency ω.

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$\{e^{i\omega t}\}$ form a complete basis in $(-\infty, \infty)$
so any function of ω or t can be expanded
in terms of them.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}\,dt = \delta(\omega-\omega')$$ orthogonality

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}\,d\omega = \delta(t-t')$$ completeness.

We always can define an antitransformed:

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{+i\omega t}\, dt$$

Multiply by $\frac{e^{-i\omega t'}}{\sqrt{2\pi}}$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t'}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\omega(t-t')}\, dt\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) \underbrace{\int_{-\infty}^{\infty} e^{-i\omega(t-t')}\, d\omega}_{2\pi\,\delta(t-t')} =$$

$$= f(t')$$

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Then

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t}\, d\omega$$

anti-transformed

3D generalization:

$$g(\bar{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\bar{r})\, e^{-i\bar{k}\cdot\bar{r}}\, d^3r$$

$$f(\bar{r}) = \frac{1}{(2\pi)^{3/2}} \int g(\bar{k})\, e^{-i\bar{k}\cdot\bar{r}}\, d^3k$$

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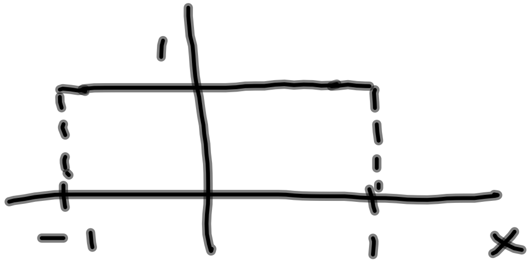
Transforming functions and calculating integrals:

$$f(x) = \begin{cases} 1 & -1 \le x \le 1 \\ 0 & |x| > 1 \end{cases}$$



$$g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\alpha x}\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-i\alpha x}}{-i\alpha} \right|_{-1}^{1} = \frac{1}{\sqrt{2\pi}} \frac{(e^{i\alpha} - e^{-i\alpha})}{i\alpha} = \frac{2}{\sqrt{2\pi}} \frac{\sin\alpha}{\alpha}$$

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What happens if we antitransform $g(\alpha)$?

Particularly for $|x|=1$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\alpha)\, e^{-i\alpha x}\, d\alpha =$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} \left(\cos\alpha x - i \sin\alpha x\right) d\alpha =$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin\alpha \cos\alpha x}{\alpha}\, d\alpha$$

For $x = \pm 1$

$$f(\pm 1) = \frac{2}{\pi} \int_0^\infty \frac{\sin\alpha \cos\alpha}{\alpha}\, d\alpha = \frac{1}{2}$$

This value interpolates at the discontinuity.

So the antitransform of a Fourier transform of a discontinue function provides the average value of the function at the discontinuity. Now

$$\int_0^\infty \frac{\sin\alpha \cos\alpha x}{\alpha}\, d\alpha = \begin{cases} \frac{\pi}{2} & \text{for } |x|<1 \\ \frac{\pi}{4} & \text{for } |x|=1 \\ 0 & \text{for } |x|>1 \end{cases}$$

We can use Fourier transforms of known functions to calculate hard to evaluate integrals.

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