11/27

Cosine and Sine transformations

Kernel for FT: $e^{-ikx} = \cos kx + i \sin kx$

Even functions are expanded only in terms of the $\cos kx$ and odd functions in terms of the $\sin kx$.

Cosine Transf.:

If $f_c(t) = f_c(-t)$ (even) then

$$g_c(\omega) = \frac{1}{\sqrt{2\pi}}\, 2\int_0^\infty f_c(t)\cos\omega t\, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(t)\cos\omega t\, dt$$

Nov 27-2:14 PM

and the AT:

$$f_c(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(\omega) \cos \omega t \, d\omega$$

Sine transformation:

if $f_s(t) = -f_s(-t)$ (odd) then

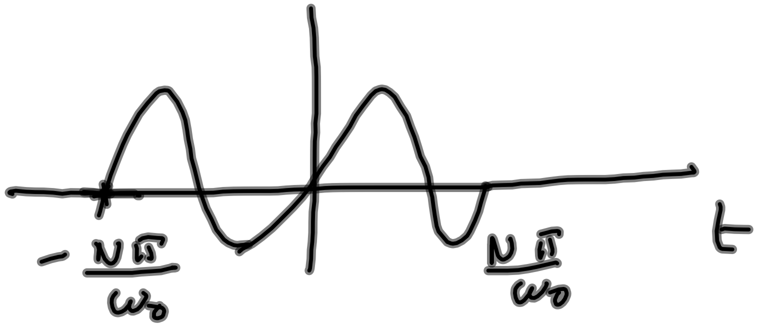
$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(t) \sin \omega t \, dt$$

and

$$f_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\omega) \sin \omega t \, d\omega$$

Nov 27-2:18 PM

Example: Spectrum of a wave packet.



$$f(t) = \begin{cases} \sin \omega_0 t & \text{if } |t| < \frac{N\pi}{\omega_0} \\ 0 & \text{if } |t| > \frac{N\pi}{\omega_0} \end{cases}$$

odd function.

Then

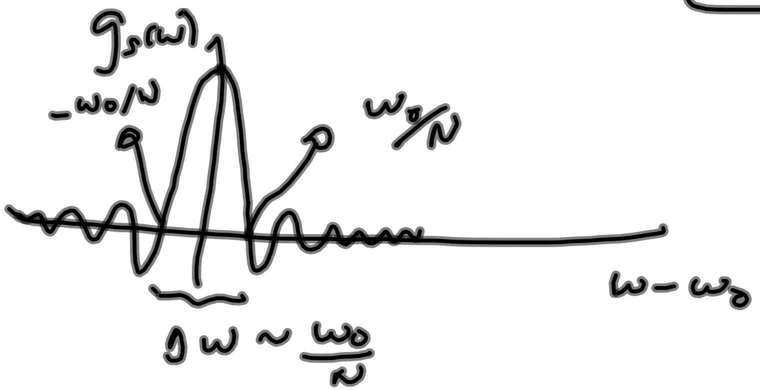
$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{N\pi}{\omega_0}} \sin \omega_0 t \, \sin \omega t \, dt = \sqrt{\frac{2}{\pi}} \left[\frac{\sin(\omega_0 - \omega)t}{2(\omega_0 - \omega)} - \frac{\sin(\omega_0 + \omega)t}{2(\omega_0 + \omega)} \right] \Bigg|_0^{\frac{N\pi}{\omega_0}} = \sqrt{\frac{2}{\pi}} \left[\frac{\sin(\omega_0 - \omega)\frac{N\pi}{\omega_0}}{2(\omega_0 - \omega)} - \frac{\sin(\omega_0 + \omega)N\pi/\omega_0}{2(\omega_0 + \omega)} \right]$$

If ω_0 is large when $\omega \sim \omega_0$ the first term is much larger than the second then:

$$g_s(\omega) \sim \sqrt{\frac{2}{\pi}} \left[\frac{\sin (\omega_0 - \omega) N\pi/\omega_0}{2(\omega_0 - \omega)} \right]$$

has zeroes when $\frac{(\omega_0 - \omega) N\pi}{\omega_0} = m\pi$

then $\boxed{\frac{(\omega_0 - \omega)}{\omega_0} = \frac{n}{N}}$ $n = \pm 1, 2, 3 \ldots$



$$\Delta\omega \Delta t = \frac{\omega_0}{N} \frac{2N\pi}{\omega_0} = 2\pi$$

If our wave packet represents photons:

$$E = \hbar \omega$$

$$\hbar \Delta \omega = \Delta E$$

then

$$\Delta E \Delta t = \hbar \Delta \omega \Delta t = \hbar 2\pi = h \quad \text{Planck's constant.}$$

The packet satisfies Heisenberg's uncertainty principle that states that

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}.$$

Nov 27-2:30 PM

Integral Transformations to solve differential equations:

- Problem in real space hard to solve.
- Problem in transformed space is easy to solve.

1) Transform.

2) Solve

3) Antitransform.

Nov 27-2:33 PM

How do derivatives transform Fourier?

Consider $f(x)$:

In k space:

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$$

What happens with $\frac{df(x)}{dx}$? We obtain:

$\lim_{x \to \pm\infty} f(x) = 0$

$$g_1(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df(x)}{dx}\, e^{ikx}\, dx = 1$$

$\int u'v\, dx = uv - \int u v'\, dx$ integration by parts.

Nov 27-2:34 PM

$$① = \frac{1}{\sqrt{2\pi}}\left[\underbrace{f(x)e^{-ikx}\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} f(x)\,\frac{de^{-ikx}}{dx}\,dx\right] =$$

$$= -\frac{ik}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx}_{g(k)} = \frac{-ik}{\sqrt{2\pi}}\,g(k) = g_1(k)$$

In general

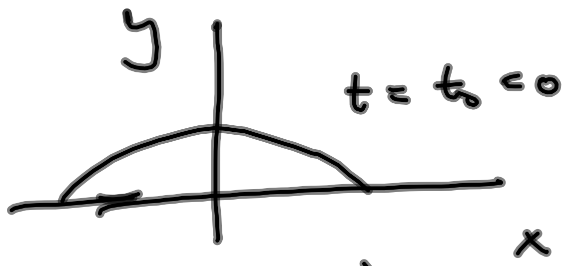
$$g_n(k) = FT\left(\frac{d^n f(x)}{dx^n}\right) = (-ik)^n\, g(k)$$

Then a differential eq. in real space becomes an algebraic eq. in Fourier space.

Nov 27-2:37 PM

Example: Wave equation for a string with fixed ends.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad (1)$$



In Fourier space:

$y = y(x,t)$

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(k,t)\, e^{-ikx}\, dk \quad (2)$$

$y = y(x, t=0) = f(x)$

with $\lim_{x\to\pm\infty} f(x) = 0.$

Plugging (2) in (1):

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -k^2 y(k,t)\, e^{-ikx}\, dk = \frac{1}{v^2}\int_{-\infty}^{\infty}\frac{\partial^2 y(k,t)}{\partial t^2}\, e^{-ikx}\, dk\, \frac{1}{\sqrt{2\pi}} \quad (3)$$

In ③ we see that for each value of k:

$$-k^2 y(k,t) = \frac{1}{v^2}\frac{\partial^2 y(k,t)}{\partial t^2}$$ harmonic oscillator eq.

then

$$y(k,t) = y_k e^{\pm i v k t}$$

For $t=0$ (initial conditions)

$$y(x,0) = f(x)$$ then $y_k = FT(f(x))$

Then

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{y_k e^{\pm i v k t}}_{y(k,t)} e^{-ikx}\, dk =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y_k e^{-ik(x\mp vt)} dk = f(x \pm vt)$$

Nov 27-2:45 PM

The exact linear combination of $f(x+vt)$ and $f(x-vt)$ is obtained from the b.c. for $y'(x,t=0)$.

Heat Flow

1

$$\frac{\partial \psi(x,t)}{\partial t} = a^2 \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

net heat out = change in heat stored

$\psi(x,t) \propto T(x,t)$ — temperature.

$$-\frac{dQ}{dt} = -c\rho\frac{dT}{dt} \qquad \bar{\nabla}_x Q =$$

$$= \bar{\nabla}_x(-\sigma \nabla_x T) = -\sigma \nabla^2 T$$

Let's write $\psi(x,t)$ in terms of k:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(k,t)\, e^{-ikx}\, dk \qquad (2)$$

Plugging (2) in (1):

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial \psi}{\partial t}(k,t)\, e^{-ikx}\, dk = \frac{a^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -k^2 \psi(k,t) e^{-ikx} dk \qquad (3)$$

Solve (3) for a fixed value of k:

$$\frac{\partial \psi}{\partial t}(k,t) = -a^2 k^2 \psi(k,t)$$

Let's solve for $\psi(k)$:

$$\underbrace{\int_{\psi(t=0)}^{\psi(t=t)} \frac{d\psi}{\psi}}_{\ln \psi(t) - \ln \psi(t=0)} = \int_0^t -a^2k^2 dt = -a^2k^2\, t \Big|_0^t = -a^2k^2 t$$

$$\ln \psi(k,t) = -a^2k^2t - \ln \psi(k,0)$$

$$\psi(k,t) = \psi(k,0)\, e^{-a^2k^2t} \qquad (3)$$

Plugging (3) in (2):

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(k,0)\, e^{-k^2a^2t}\, e^{-ikx}\, dk$$

Now we need to use b.c.'s:

Assume that $\psi(k,0) = C$ (independent of k).

in x space it would be $\psi(x,0) = c\,\delta(x)$.

$$\psi(x,t) = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2a^2t - ikx}\, dk =$$

complete the square.

$$= \frac{c}{a\sqrt{2t}}\, e^{-x^2/4a^2t}.$$

Green functions for inhomogeneous Helmholz equation:

$$\nabla_x^2 \psi(\bar{x}) + k^2 \psi(\bar{x}) = -4\pi f(\bar{x}) \quad (1)$$

Helmolz eq.

$f(\bar{x})$ → source of the perturbation $\psi(\bar{x})$.

$\psi(x)$ can be a wave shape and $f(\bar{x})$ is an impulse.

$\psi(x)$ can be a potential and $f(\bar{x})$ is the charge distribution.

We know that

$\nabla^2 G(\bar{x}) = -4\pi\,\delta(\bar{x})$ for charge at $\bar{x}'=0$

or

$\nabla^2 G(\bar{x},\bar{x}') = -4\pi\,\delta(\bar{x}-\bar{x}')$ for $\bar{x}'\neq 0$.

It also can be shown that

$\nabla^2 G(\bar{x},\bar{x}') + \lambda\, q(\bar{x})\, G(\bar{x},\bar{x}') = -4\pi\,\delta(\bar{x}-\bar{x}')$ satisfies that

$\psi(\bar{x}) = \frac{1}{4\pi}\int_{\text{all space}} f(\bar{x}')\, G(\bar{x},\bar{x}')\, d^3x'$ (no surface terms).

We are going to find $G(\bar{x}, \bar{x}')$ for Helmholtz eq. with $f(\bar{x}') = \delta(\bar{x} - \bar{x}')$ and we are going to use $G(\bar{x}, \bar{x}')$ to obtain $\psi(\bar{x})$ for arbitrary $f(\bar{x}')$.

We need to find $G(\bar{x}, \bar{x}')$ such that:

$$\nabla^2 G + k^2 G = -4\pi\, \delta(\bar{x} - \bar{x}') \quad (3)$$

If I work in infinite isotropic space then $G(\bar{x}, \bar{x}')$ cannot depend on θ, φ, θ' or φ' then

$$G(\bar{x}, \bar{x}') = G(R) = G(r - r') \quad (4)$$

Nov 27-3:12 PM

We work in spherical coordinates. Plugging (4) in (3) we get:

$$\frac{1}{R}\frac{d^2}{dR^2}(RG) + k^2 G(R) = -4\pi\,\delta(R) \qquad (5)$$

For the homogeneous eq. we find that

$$RG = A e^{ikR} + B e^{-ikR} \qquad \text{i.e.} \qquad (6)$$

(6) solves the homogeneous version of (5) then

$$G(R) = \frac{A e^{ikR} + B e^{-ikR}}{R} \qquad (7)$$

For $k \to 0$ (5) becomes Poisson and $G(R)$ given by (7)

Nov 27-3:16 PM

becomes $\frac{1}{R}$ then $A+B=1$.

Let's propose:

$$\boxed{G_k^{\pm}(R) = \frac{e^{\pm ikR}}{R}}$$

So

$$G_k^{\pm}(|\bar{r}-\bar{r}'|) = \frac{e^{\pm ik|r-r'|}}{|r-r'|}$$

Now we can solve ②:

$$\boxed{\psi(r) = \frac{1}{4\pi}\int_{\text{all space}} dr' \, \frac{e^{\pm ik|r-r'|}}{|r-r'|} f(r')}$$