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Reminder: Fill class evaluation and send me an e-mail.

Inhomogeneous Wave Equation

$$\nabla^2 \psi(\bar{x},t) - \frac{1}{c^2}\frac{\partial^2 \psi(\bar{x},t)}{\partial t^2} = -4\pi f(\bar{x},t) \quad (1)$$

We can obtain $\psi(\bar{x},t)$ if we know the Green function $G(\bar{x},\bar{x}',t,t')$ then:

$$\psi(\bar{x},t) = \frac{1}{4\pi}\int d^3x' \int dt' \, G(\bar{x},\bar{x}',t,t') f(\bar{x}',t') \quad (2)$$

no surface terms since $\int$ is over all space.

G is a function that solves:

$$\nabla^2 G(\bar{x}, \bar{x}', t, t') - \frac{1}{c^2}\frac{\partial^2 G(\bar{x}, \bar{x}', t, t')}{\partial t^2} = -4\pi\, \delta(\bar{x} - \bar{x}')\, \delta(t - t') \qquad (3)$$

To simplify the $\frac{\partial^2}{\partial t^2}$ term I'm going to Fourier transform the variable t:

Then

$$\psi(\bar{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(\bar{x}, \omega)\, e^{-i\omega t}\, d\omega \qquad (4)$$

$$f(\bar{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\bar{x}, \omega)\, e^{-i\omega t}\, d\omega \qquad (5)$$

Plugging 4 and 3 in 1 we obtain:

$$\nabla^2 \psi(\bar{x},\omega) + \overbrace{\frac{\omega^2}{c^2}}^{k^2} \psi(\bar{x},\omega) = -4\pi f(\bar{x},\omega)$$

Helmholz equation!

for each value of ω

We know how to find $\psi(\bar{x},\omega)$ since we did it last time:

$$\psi(\bar{x},\omega) = \frac{1}{4\pi}\int d^3x' \, \underbrace{G(\bar{x},\bar{x}',\omega)}_{G_{k=\pm\frac{\omega}{c}}(\bar{x},\bar{x}')} f(\bar{x}',\omega)$$

We found that

$$G(\bar{x},\bar{x}',\omega) = G_k(\bar{x},\bar{x}') = \frac{e^{\pm ik|\bar{x}-\bar{x}'|}}{|\bar{x}-\bar{x}'|}$$

with $k^2 = \omega^2/c^2$

Next we need to find $G(\bar{x},\bar{x}', t, t')$:

Let's FT in t Eq (3):

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\nabla^2 G(\bar{x},\bar{x}',\omega,t')\, e^{-i\omega t}\, d\omega - \frac{1}{c^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial t^2}\left[G(\bar{x},\bar{y}',\omega,t')\, e^{-i\omega t}\right] d\omega =$$

$$= -4\pi\,\delta(\bar{y}-\bar{x}')\,\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\, e^{i\omega t'}\, d\omega}_{\delta(t-t')}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\nabla^2 G(\bar{x},\bar{x}',\omega,t') + \frac{\omega^2}{c^2} G(\bar{x},\bar{x}',\omega,t') \right) e^{-i\omega t} d\omega =$$

$$= -4\pi\, \delta(\bar{x}-\bar{x}') \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t'} e^{-i\omega t} d\omega \quad (6)$$

If $G(\bar{x},\bar{x}',\omega,t') = G(\bar{x},\bar{x}',\omega)\, g(t') \quad (7)$

Plugging (7) in (6) we find that

$$\nabla^2 G(\bar{x},\bar{x}',\omega)\, g(t') + \frac{\omega^2}{c^2} G(\bar{x},\bar{x}',\omega)\, g(t') =$$

$$= -4\pi\, \delta(\bar{x}-\bar{x}') \frac{1}{\sqrt{2\pi}} e^{i\omega t'}$$

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Since

$$\nabla^2 G(\bar{x}, \bar{x}', \omega) + \frac{\omega^2}{c^2} G(\bar{x}, \bar{x}', \omega) = -4\pi\, \delta(\bar{x} - \bar{x}')$$

we see that

$$g(t') = \frac{e^{-i\omega t'}}{\sqrt{2\pi}}$$

Then:

$$G(\bar{x}, \bar{x}', \omega, t') = G(\bar{x}, \bar{x}', \omega) \frac{e^{-i\omega t'}}{\sqrt{2\pi}} =$$

$$= \frac{e^{\pm ik|\bar{x} - \bar{x}'|}}{|\bar{x} - \bar{x}'|} \frac{e^{-i\omega t'}}{\sqrt{2\pi}}$$

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Now we need to antitransform $G(\bar{x}, \bar{x}', \omega, t')$ to $G(\bar{x}, \bar{x}', t, t')$:

$$G^{\pm}(\bar{x}, \bar{x}', t, t') = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\bar{x}-\bar{x}'|}}{|\bar{x}-\bar{x}'|} e^{i\omega t'} e^{-i\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\left[t' - t \pm \frac{|\bar{x}-\bar{x}'|}{c}\right]}}{|\bar{x}-\bar{x}'|} d\omega =$$

$$= \frac{1}{\cancel{2\pi}\,|\bar{x}-\bar{x}'|} \delta\left(t' - \left[t \mp \frac{|\bar{x}-\bar{x}'|}{c}\right]\right) \cancel{2\pi} =$$

since $\int_{-\infty}^{\infty} e^{i\omega(t-t')} d\omega = 2\pi\, \delta(t-t')$

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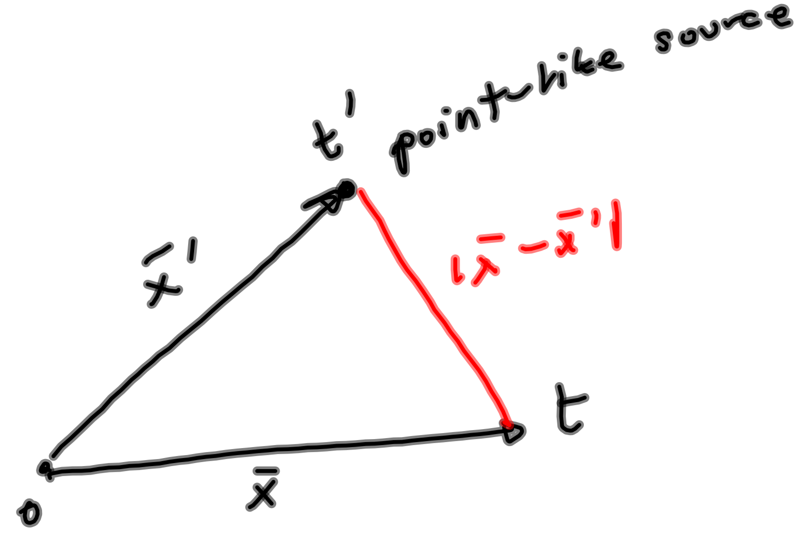
$$= \frac{\delta\left(t' - \left[t \mp \frac{|\bar{x}-\bar{x}'|}{c}\right]\right)}{|\bar{x}-\bar{x}'|} \equiv G^{\pm}(\bar{x},\bar{x}',t,t')$$



it takes a time $t-t'$ for the effect of the source at $\bar{x}'$ to reach $\bar{x}$.

If $t'<t$

$$t' = t - \frac{|\bar{x}-\bar{x}'|}{c}$$

$$t = t' + \frac{|\bar{x}-\bar{x}'|}{c}$$

$G^{+}(\bar{x},\bar{x}',t,t')$ reproduces this situation. It is called the retarded Green function.

$G^-(\bar{x},\bar{x}',t,t')$ is the advanced Green function.

Is needed for certain b.c. but G^+ is most commonly needed,

Examples:

At $t \to -\infty$ there is a wave $\psi_{in}(\bar{x},t)$ that satisfies $\nabla^2 \psi_{in} - \frac{1}{c^2}\frac{\partial^2 \psi_{in}}{\partial t^2} = 0$

$\psi_{in} \propto e^{-i(\bar{k}\cdot\bar{x}-\omega t)}$. Now at time t_0 you turn a source $j(\bar{x},t)$

$$\psi(\bar{x},t) = \psi_{in}(\bar{x},t) + \frac{1}{4\pi}\iint \overbrace{G^+(\bar{x},t,\bar{x}',t')\, j(\bar{x}',t')}^{0 \text{ for } t<t'}\, d^3x'\, dt'$$

If at $t \to \infty$ there is $\psi_{out}(\bar{x}, t)$ that satisfies the homogeneous wave eq. after a source $f(\bar{x}, t)$ has been shut down then:

$$\psi(\bar{x}, t) = \psi_{out}(\bar{x}, t) + \frac{1}{4\pi} \iint_{-\infty}^{\infty} \underbrace{G^{-}(\bar{x}, t, \bar{x}', t')}_{\text{it cuts the time } \int \text{ at } t = t_0 \text{ when the source is shut down}} \times f(\bar{x}', t')\, d^3x'\, dt'$$

all space

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Helmholz equation

solution in arbitrary basis:

$$\nabla^2 \psi(\bar{r}) + k^2 \psi(\bar{r}) = \rho(\bar{r}) \quad (1)$$

$$\psi(\bar{r}) = \frac{1}{4\pi} \int \rho(\bar{r}') \, G(\bar{r}, \bar{r}') \, d\bar{r}' + \text{surface terms} \quad (2)$$

(appear if V is finite)

Let's find $G(\bar{r}, \bar{r}')$.

$G(\bar{r}, \bar{r}')$ is solution to

$$\nabla^2 G(\bar{r}, \bar{r}') + k^2 G(\bar{r}, \bar{r}') = \delta(\bar{r} - \bar{r}') \quad (3)$$

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Assume that $\{\phi_n(\bar{r})\}$ are a set of orthonormal functions each of whom solve the homogeneous version of ①:

$$\nabla^2 \phi_n(\bar{r}) + k_n^2 \phi_n(\bar{r}) = 0 \quad (4)$$

Notice that ϕ_n can be Bessel functions, cosines, etc.

Now

$$\psi^{homo}(\bar{r}) = \sum_n A_n \phi_n(\bar{r}) \quad (5)$$

solution of ① homogeneous with index k

Also

$$G(\bar{r}, \bar{r}') = \sum_{n=0}^{\infty} a_n(r') \, \phi_n(\bar{r}) \qquad (6)$$

Also

$$\delta(\bar{r} - \bar{r}') = \sum_{m=0}^{\infty} \phi_m^*(\bar{r}) \, \phi_m(\bar{r}') \qquad \text{completitude} \quad (7)$$

Plugging (7), (6) in (3):

$$\nabla^2 \left[\sum_{m=0}^{\infty} a_n(\bar{r}') \, \phi_m(\bar{r}) \right] + k^2 \sum_{m=0}^{\infty} a_m(\bar{r}') \, \phi_m(\bar{r}) =$$

$$= \sum_{m=0}^{\infty} \phi_m^*(\bar{r}) \, \phi_m(\bar{r}') \qquad (8)$$

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From (5) we see that

$$\nabla^2 \phi_n(\bar{r}) = -k_n^2\, \phi_n(\bar{r}) \quad \text{then}$$

$$\sum_{n=0}^{\infty} a_n(\bar{r}')\,(-k_n^2)\,\phi_n(\bar{r}) + k^2 \sum_n a_n(\bar{r}')\,\phi_n(\bar{r}) = \sum_n \phi_n^*(\bar{r}')\,\phi_n(\bar{r})$$

From orthogonality of the $\{\phi_n(\bar{r})$

$$\boxed{a_n(\bar{r}') = \frac{\phi_n^*(\bar{r}')}{k^2 - k_n^2}}$$

$$\therefore \boxed{G(\bar{r},\bar{r}') = \sum_{n=0}^{\infty} \frac{\phi_n^*(\bar{r}')\,\phi_n(\bar{r})}{k^2 - k_n^2}} \quad (9)$$

Let's compare (9) with our previous solution of Helmholz eq. in isotropic space.

$$\{\phi_n(\bar{r})\} \longrightarrow \{e^{ikx}\}$$

We had

$$(\nabla^2 + k^2)\, G(x,x') = \delta(x-x') \qquad (1)$$

$$\nabla^2 e^{ikx} + k^2 e^{ikx} = 0$$

they solve the homogeneous wave eq.

$$\psi^{homo}(x) = \frac{1}{\sqrt{2\pi}} \int a_k e^{ikx}\, dk$$

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$$G(x, x') = \int a_k(x')\, e^{ikx}\, dk \quad (2)$$

$$\delta(x - x') = \frac{1}{2\pi} \int e^{i(x-x')k}\, dk \quad (3)$$

Plugging (2), (3) in (1):

$$(\nabla^2 + k^2) \int a_{k'}(x')\, e^{ik'x}\, dk' = \frac{1}{2\pi} \int e^{i(x-x')k'}\, dk'$$

$$\int \left(-a_{k'}(x')\, k'^2 e^{ik'x} + k^2 a_{k'}(x')\, e^{ik'x} \right) dk' =$$

$$= \frac{1}{2\pi} \int e^{i(x-x')k'}\, dk'$$

$$\int a_{k'}(x')\, (k^2 - k'^2)\, e^{ik'x}\, dk' = \frac{1}{2\pi} \int e^{i(x-x')k'}\, dk'$$

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Then

$$a_{k'}(x')\,(k^2 - k'^2) = \frac{e^{-ik'x'}}{2\pi}$$

$$\therefore\ a_{k'}(x') = \frac{e^{-ik'x'}}{2\pi(k^2 - k'^2)} \qquad (4)$$

(4) in (2):

$$G(x, x') = \frac{1}{2\pi}\int \frac{e^{-ik'x'}}{k^2 - k'^2}\, e^{ik'x}\, dk' =$$

$$= \frac{1}{2\pi}\int \frac{e^{-ik'(x'-x)}}{k^2 - k'^2}\, dk' = \frac{e^{ik|x-x'|}}{4\pi\,|x-x'|}.$$

as before

can be done
(9.128 in the book)