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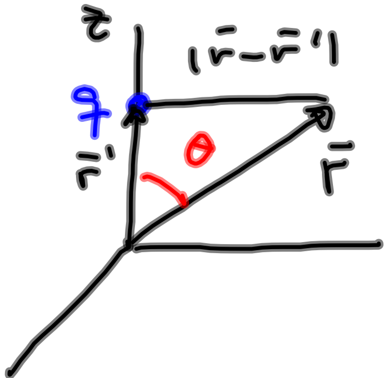
$\frac{1}{|\bar{r}-\bar{r}'|}$ in terms of $Y_{\ell m}(\theta, \varphi)$

We know

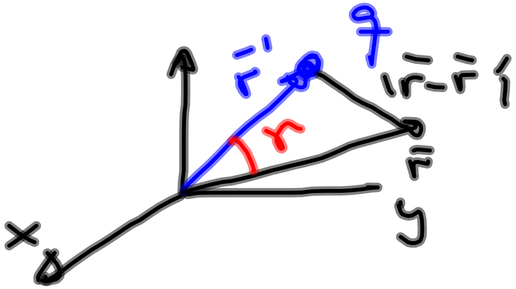


if $q = 4\pi\varepsilon_0$

$$\phi_q(\bar{r}) = \frac{1}{|\bar{r}-\bar{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

Now:

$$\frac{1}{|\bar{r}-\bar{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\gamma) \qquad (1)$$



$\bar{r} = (r, \theta, \varphi)$

$\bar{r}' = (r', \theta', \varphi')$

Theorem of addition of Spherical Harmonics

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\varphi)\, Y_\ell^{m*}(\theta',\varphi') \quad (2)$$

Plugging (2) in (1) we obtain:

$$\boxed{\frac{1}{|\vec r-\vec r\,'|} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{r_<^{\ell}}{r_>^{\ell+1}}\,\frac{1}{2\ell+1}\,Y_\ell^{m*}(\theta',\varphi')\,Y_\ell^m(\theta,\varphi)}$$

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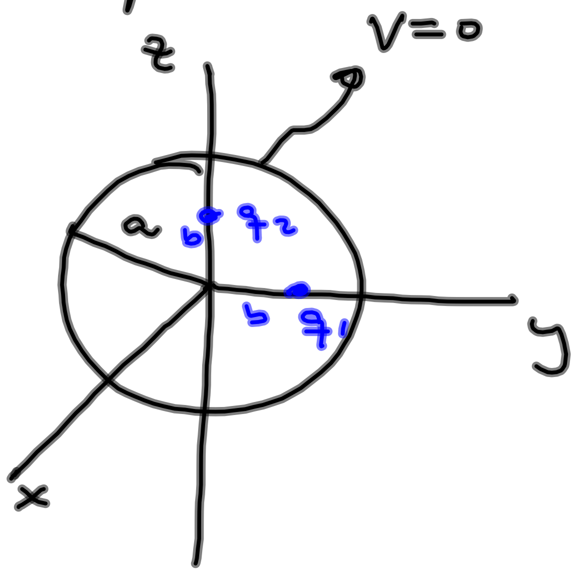
Example:



$\phi(r,\theta,\varphi) = ?$ for $r \leq a$.

Use the superposition principle:

For q_1 $r' = b$, $\varphi' = \frac{\pi}{2}$, $\theta' = \frac{\pi}{2}$

For q_2 $r' = b$, $\theta' = 0$, φ' (undefined but irrelevant since $m=0$)

$$\phi(\bar{r}) = \phi_{shell} + \phi_{q_1} + \phi_{q_2}$$

$$\phi_{shell} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell r^\ell Y_{\ell m}(\theta,\varphi)$$

$$\phi_{q_1} = \frac{4\pi q_1}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{r_<^\ell}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_{\ell m}(\theta,\varphi)\, Y_{\ell m}^*\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\Phi_{q_2} = \frac{4\pi q_2}{4\pi \varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell 0}(0,\varphi')$$

Here $r_>$ ($r_<$) is the larger (smaller) between r and $r'=b$.

Find A_ℓ. We know that at $r=a$

$\Phi(r,\theta,\varphi)=0$.

Then:

$$0 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_\ell a^\ell + \frac{1}{(2\ell+1)\varepsilon_0} \frac{b^\ell}{a^{\ell+1}} \left[q_2 \overbrace{\underbrace{Y_\ell^{0*}(0,\varphi')}_{\sqrt{\frac{2\ell+1}{4\pi}}}}^{\sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell(\overset{\cos 0}{1})} + q_1 Y_\ell^{*m}\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right) \right] \right] Y_\ell^m(\theta,\varphi)$$

$$A_\ell = \frac{1}{\varepsilon_0}\frac{b^\ell}{a^{2\ell+1}}\frac{1}{2\ell+1}\left[q_2\, Y_\ell^{m*}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) + q_1\sqrt{\frac{2\ell+1}{4\pi}}\right]$$

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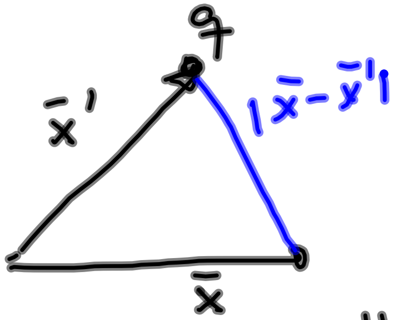
Inhomogeneous differential equations:

Poisson's equation

Green function method (Ch. 9 in book)

Also see Physics Today Dec. 2003 page 41.

$$\nabla^2 \phi = -\frac{\rho(\bar{x})}{\varepsilon_0} \quad (1)$$ Poisson's eq.



if $q = 4\pi\varepsilon_0$

$$\phi_q(\bar{x}) = \frac{1}{|\bar{x}-\bar{x}'|} \quad (2)$$

Here: $\rho(\bar{x}) = q\,\delta(\bar{x}-\bar{x}') = 4\pi\varepsilon_0\,\delta(\bar{x}-\bar{x}') \quad (3)$

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Replacing ② and ③ in ① we obtain:

$$\boxed{\nabla^2\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) = -\frac{4\pi\,\varepsilon_0\,\delta(\bar{x}-\bar{x}')}{\varepsilon_0} = -4\pi\,\delta(\bar{x}-\bar{x}')}$$

We also know that the potential $\phi(\bar{x})$ due to a distribution of charge $\rho(\bar{x})$ in all space is given by:

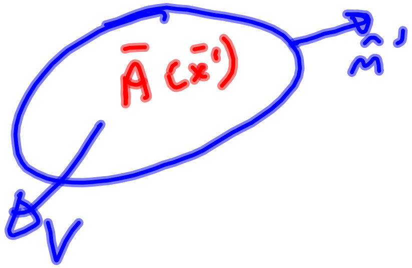
$$\phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{\rho(\bar{x}')\,d^3x'}{|\bar{x}-\bar{x}'|}$$

What happens if we want to find $\phi(\bar{x})$ due to $\rho(\bar{x}')$ in a volume V?

Mathematical detour: Green's theorem.

Consider $\bar{A}(\bar{x}')$ a vector field. Then

$$① \int_V \bar{\nabla} \cdot \bar{A} \, d^3x' = \oint_S \bar{A} \cdot \hat{n}' da'$$ divergence theorem



Assume that

$$\bar{A}(\bar{x}') = \phi(\bar{x}') \, \bar{\nabla} \psi(\bar{x}') \quad ②$$

ϕ and ψ are scalar functions.

Let's plug (2) in (1) :

$$\bar{\nabla}\cdot\bar{A} = \bar{\nabla}(\phi\,\bar{\nabla}\psi) = \bar{\nabla}\phi\;\bar{\nabla}\bar{\psi} + \phi\;\nabla^2\psi$$

$$\bar{A}\cdot\hat{m} = \phi\;\bar{\nabla}\psi\cdot\hat{m}' = \phi\,\frac{\partial\psi}{\partial m'}$$

$$\int_V (\phi\;\nabla^2\psi + \bar{\nabla}\phi\;\bar{\nabla}\psi)\,d^3x' = \oint_S \phi\,\frac{\partial\psi}{\partial m'}\,da' \qquad (3)$$

Now take

$\bar{A}(\bar{x}') = \psi\;\bar{\nabla}\phi$ and plug it in (1) :

$$\int_V (\psi\;\nabla^2\phi + \bar{\nabla}\psi\;\bar{\nabla}\phi)\,d^3x' = \oint_S \psi\,\frac{\partial\phi}{\partial m'}\,da' \qquad (4)$$

Do 3 - 9 :

$$\boxed{\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi \right) d^3x' = \oint_S \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right) da'}$$

6 Green's theorem

Now we will use 6 to address the question of finding $\phi(\bar{x})$ due to $\rho(\bar{x})$ in a finite volume V.

Assume that $\phi(\bar{x})$ is the electrical potential.

Let's $\psi = \frac{1}{|\bar{x}-\bar{x}'|}$ potential due to charge $4\pi\varepsilon_0$ at $\bar{x}'$.

Putting ϕ and ψ in ⑥:

$$\int_V \Big[\phi(\bar{x}')\underbrace{\left(-4\pi\,\delta(\bar{x}-\bar{x}')\right)}_{\nabla^2\psi} + \underbrace{\frac{1}{|\bar{x}-\bar{x}'|}}_{\psi}\,\underbrace{\left(\frac{-\rho(\bar{x}')}{\varepsilon_0}\right)}_{\nabla^2\phi}\Big]\,d^3x' =$$

$$= \oint_S \left[\phi(\bar{x}')\frac{\partial}{\partial n'}\left[\frac{1}{|\bar{x}-\bar{x}'|}\right] - \frac{1}{|\bar{x}-\bar{x}'|}\frac{\partial\phi}{\partial n'}\right]da'$$

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$$-4\pi \int \phi(\bar{x}')\,\delta(\bar{x}-\bar{x}')\,d^3x' = \phi(\bar{x}) \text{ if } \bar{x} \in V$$

0 if $\bar{x}$ outside V.

$$-4\pi\,\phi(\bar{x}) + \frac{1}{\varepsilon_0}\int_V \frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|}\,d^3x' = \oint_S \left[\phi \frac{\partial}{\partial n'}\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) - \frac{1}{|\bar{x}-\bar{x}'|}\frac{\partial \phi}{\partial n'}\right] da' \quad \text{if } \bar{x} \in V.$$

Then for $\bar{x} \in V$:

$$\phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\bar{x}')\,d^3x'}{|\bar{x}-\bar{x}'|} + \frac{1}{4\pi}\oint_S \left[\frac{1}{|\bar{x}-\bar{x}'|}\frac{\partial \phi}{\partial n'} - \phi \frac{\partial}{\partial n'}\left[\frac{1}{|\bar{x}-\bar{x}'|}\right]\right] da'$$

We see that if $V \to \infty$ the surface integrals vanish because $\frac{1}{|\bar{x}-\bar{x}'|} \to 0$ and we obtain

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{all space}} \frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|} d^3x'$$ that we already know.

If V is finite, we need to provide b.c. on the surface: (two types)

1) Dirichlet b.c.: provide Φ on surface.

2) Von Neumann b.c: provide $\left.\frac{\partial \Phi}{\partial n'}\right|_S = E_n|_S$

To satisfy the b.c. we need to consider

$$\psi(x') = \frac{1}{|\bar{x}-\bar{x}'|} + F(\bar{x},\bar{x}') = \underbrace{G(\bar{x},\bar{x}')}_{\text{Green function.}}$$

$F(\bar{x},\bar{x}')$ has to be such that

$$\nabla^2 F(\bar{x},\bar{x}') = 0 \quad \text{inside } V$$

so that

$$\nabla^2 G = \underbrace{\nabla^2\left(\frac{1}{|\bar{x}-\bar{x}'|}\right)}_{-4\pi\,\delta(\bar{x}-\bar{x}')} + \underbrace{\nabla^2 F}_{0} = -4\pi\,\delta(\bar{x}-\bar{x}') \quad \text{in } V.$$

Replacing ψ by G in Green's theorem gives us:

$$\int_V \Big(\phi \underbrace{\nabla^2 G}_{-4\pi\delta(\bar{x}-\bar{x}')} - G \underbrace{\nabla^2 \phi}_{-\frac{\rho(\bar{x}')}{\varepsilon_0}}\Big)\, d^3x' = \oint_S \left(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'}\right) da'$$

$$\boxed{\phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\bar{x}')\, G(\bar{x},\bar{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \left(G\frac{\partial \phi}{\partial n'} - \phi\frac{\partial G}{\partial n'}\right) da'}$$

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1) Dirichlet: we know ϕ_S then we chose $G_D(\bar{x}, \bar{x}')$ so that $G|_S = 0$ to cancel $\frac{\partial \phi}{\partial n'}$ surface term.

2) von Neumann

we find $G(\bar{x}, \bar{x}')$ so that $\left.\frac{\partial G_N}{\partial n'}\right|_S = -\frac{4\pi}{S}$

vanishes for $S \to \infty$

Next time: we will see that

$G_D = \frac{1}{|\bar{x} - \bar{x}'|} + F(\bar{x}, \bar{x}') \Rightarrow$ potential of a point charge in V plus potential of charges outside V so that $G|_S = 0$.

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