Green Functions

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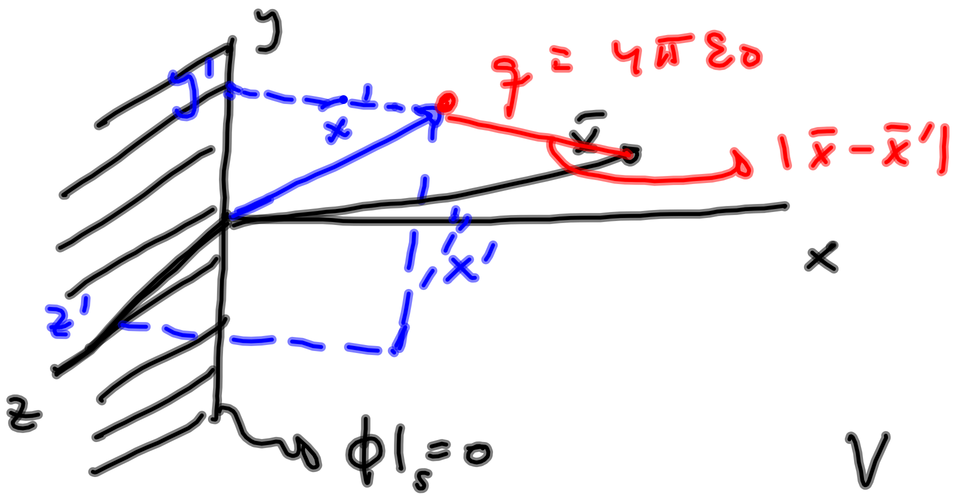
Example :

Find $G(\bar{x}, \bar{x}')$ with Dirichlet b.c. for V given by $x \geq 0$, $-\infty \leq y \leq \infty$, $-\infty \leq z \leq \infty$



$$G(\bar{x}, \bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} + F(\bar{x}, \bar{x}')$$

Since $\nabla^2 F = 0$ in V F has to be the potential of charge outside V so that $G(x, x')|_s = 0$

$$G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} + \frac{q'}{|\bar{x}-\bar{x}_0|\,4\pi\varepsilon_0}$$

Find q' and $\bar{x}_0$.

If we chose $q' = -4\pi\varepsilon_0$

Then we need that on the surface, for $x=0$,

$$\left.\frac{1}{|\bar{x}-\bar{x}'|}\right|_S = \left.\frac{1}{|\bar{x}-\bar{x}_0|}\right|_S \quad \text{since } G(\bar{x},\bar{x}')\big|_S = 0.$$

$$|\bar{x}-\bar{x}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

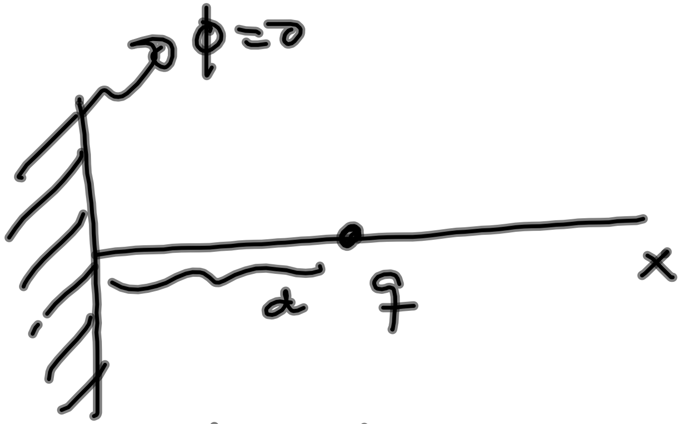
$$|\bar{x}-\bar{x}_0| = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$

At $x=0$ $|\bar{x}-\bar{x}_0| = |\bar{x}-\bar{x}'|$ if $x_0 = -x'$, $y_0 = y'$, $z_0 = z'$.

Then

$$\boxed{G(\bar{x},\bar{x}') = \frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}} - \frac{1}{\left[(x+x')^2+(y-y')^2+(z-z')^2\right]^{1/2}}}$$

Now solve this "well-known" problem:



Find $\phi(\bar{x})$ in V:

$$\phi(\bar{x}) = \int_V \rho(\bar{x}')\, G(\bar{x}, \bar{x}')\, d^3x' - \frac{1}{4\pi} \oint_S \phi_s \frac{\partial G}{\partial n'} da'$$

In this case $\rho(\bar{x}') = q\, \delta(x'-d)\, \delta(y')\, \delta(z')$

$\phi_s = 0$

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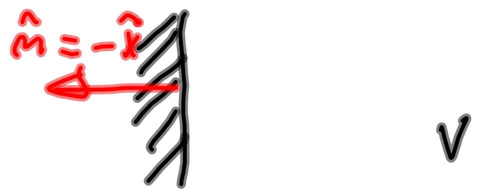
$$\Phi(\vec{x}) = \int_0^\infty dx' \int_{-\infty}^\infty dz' \int_{-\infty}^\infty dy' \; q\, \delta(x'-d)\, \delta(y')\, \delta(z')$$

$$\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}} \right]$$

$$= \frac{q}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{q}{\sqrt{(x+d)^2+y^2+z^2}}$$

If $\Phi_S \neq 0$ you need to calculate $\left.\frac{\partial G}{\partial n'}\right|_S$



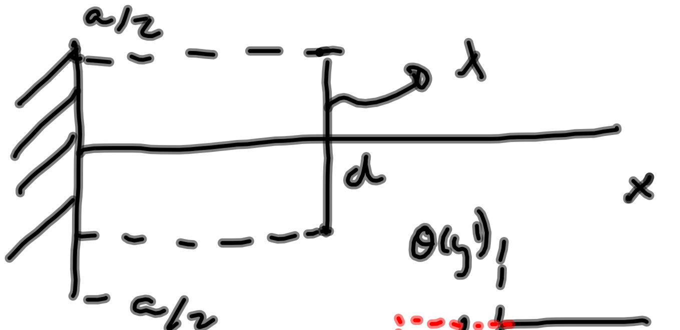
$$\frac{\partial G}{\partial n'} = \frac{\partial G}{\partial(-x')}$$

If $\phi_s = V$ then you can show:

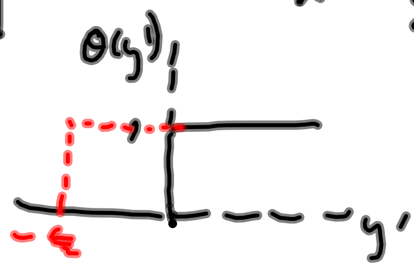
$$\frac{V}{4\pi}\int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \left.\frac{\partial G}{\partial(-x')}\right|_{x'=0} = V$$

OK because the principle of superposition is valid.

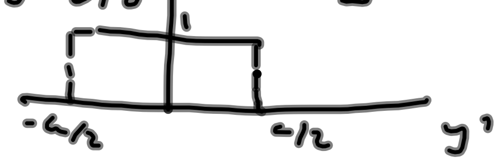
$$\rho(\bar{x}') = \lambda\,\delta(x'-d) \times \theta\left(y'-\frac{a}{2}\right)\left[1-\theta\left(y'+\frac{a}{2}\right)\right]$$



Step function $\theta(y') = \begin{cases} 0 & \text{if } y' < 0 \\ 1 & \text{if } y' > 0 \end{cases}$

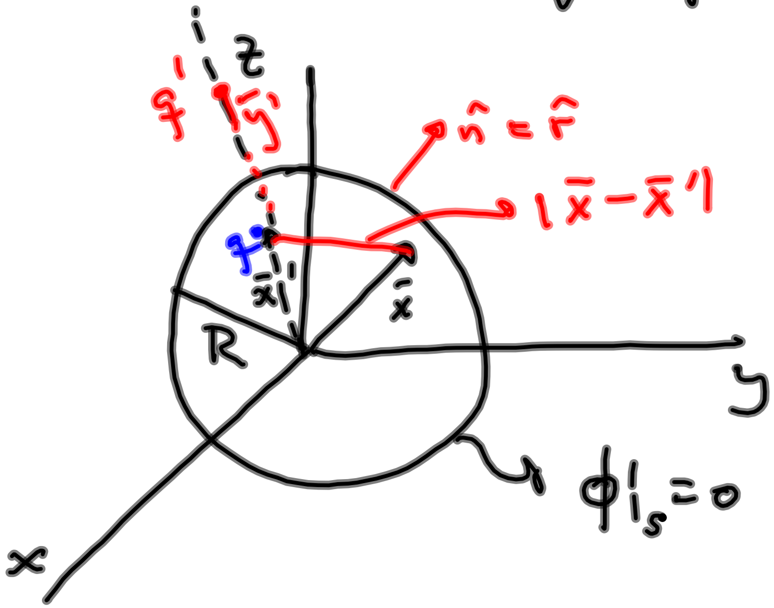


$\theta(y'-\frac{a}{2})[1-\theta(y'+\frac{a}{2})]$



Green function for spherical volume.

V defined by $r < R$ (inside the sphere)



$q = 4\pi\varepsilon_0$

$$G(\bar{x}, \bar{x}') = \frac{1}{|\bar{x} - \bar{x}'|} + F(\bar{x}, \bar{x}')$$

F is the potential of an external charge that makes $G(\bar{x}, \bar{x}')\big|_R = 0$.

$$F(\bar{x}, \bar{x}') = \frac{q'}{4\pi\varepsilon_0 \, |\bar{x} - \bar{y}'|}$$

we need to find q' and y' in terms of x' and R.

At $r = R \qquad G(\bar{r}, \bar{r}') = 0$.

$$y' > R$$

$$0 = \frac{1}{|\bar{x} - \bar{x}'|}\Bigg|_{x=R} + \frac{q'}{4\pi\varepsilon_0 |\bar{x} - \bar{y}'|_{x=R}}$$

$$0 = \frac{1}{|x\hat{n} - x'\hat{n}'|}\Bigg|_{x=R} + \frac{q'}{4\pi\varepsilon_0 |x\hat{n} - y'\hat{n}'|}\Bigg|_{x=R}$$

$$0 = \frac{1}{|R\hat{n} - x'\hat{n}'|} + \frac{q'}{4\pi\varepsilon_0 |R\hat{n} - y'\hat{n}'|}$$

$$0 = \frac{1}{R|\hat{n} - \frac{x'}{R}\hat{n}'|} + \frac{q'}{4\pi\varepsilon_0 y' |\frac{R}{y'}\hat{n} - \hat{n}'|}$$

Let's request that:

$$\frac{1}{R} = -\frac{q'}{4\pi\varepsilon_0 y'} \qquad (1)$$

and

$$\left|\hat{n} - \frac{x'}{R}\hat{n}'\right| = \left|\frac{R}{y'}\hat{n} - \hat{n}'\right|$$

$\hat{n}.\hat{n} = 1$

$\hat{n}'.\hat{n}' = 1$

$$\cancel{\underbrace{\hat{n}.\hat{n}}_{1}} - 2\frac{x'}{R}\hat{n}.\hat{n}' + \frac{x'^2}{R^2}\underbrace{\hat{n}'.\hat{n}'}_{1} = \frac{R^2}{y'^2}\underbrace{\hat{n}.\hat{n}}_{1} - 2\frac{R}{y'}\hat{n}.\hat{n}' + \cancel{\underbrace{\hat{n}'.\hat{n}'}_{1}}$$

$$\frac{2x'}{R} = \frac{2R}{y'} \Rightarrow \boxed{y' = \frac{R^2}{x'}}$$

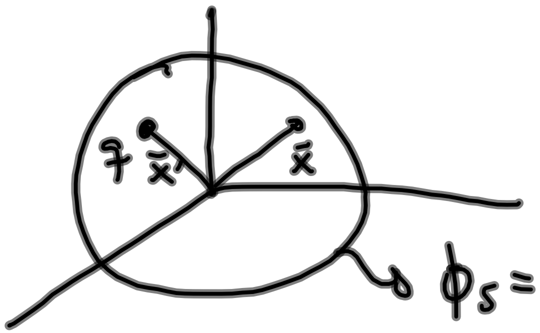
and $\dfrac{x'^2}{R^2} = \dfrac{R^2}{y'^2}$ (2)

Plugging (2) in (1) we get

$$\boxed{q'} = -\frac{4\pi\varepsilon_0}{R\,x'}R^2 = \boxed{-\frac{R\,4\pi\varepsilon_0}{x'}} \qquad (3)$$

Then:

$$\boxed{G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} - \frac{R}{x'\left|\bar{x} - R^2\frac{\hat{n}'}{x'}\right|}} \qquad (4)$$



$q = 4\pi\varepsilon_0$

$\phi_s = 0$

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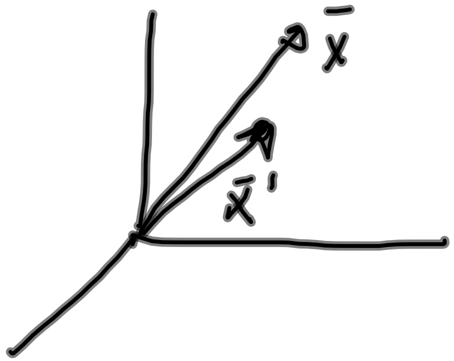
It is useful to write 4 in terms of

$Y_{\ell m}(\theta, \varphi)$ - $\frac{1}{|\bar{x}-\bar{x}'|}$ is easy since you

already have the expression -

If $\bar{x} = (r, \theta, \varphi)$ $\bar{x}' = (r', \theta, \varphi)$ $\Bigg\}$ you'll get 2 expressions one for $r_< = x'$ $r_> = x$ another for $r_> = x$, $r_< x'$



The terms $\frac{R}{x'} \frac{1}{\left|\bar{x} - \frac{R^2 \hat{n}'}{x'}\right|}$

is expanded as $\frac{1}{|\bar{x} - \bar{x}'|}$ with $\bar{x}' \equiv \frac{R^2}{x'} \hat{n}'$

Notice that $\frac{R^2}{x'} > x$ because $x < R$

so your expansion is not going to have 2 pieces.

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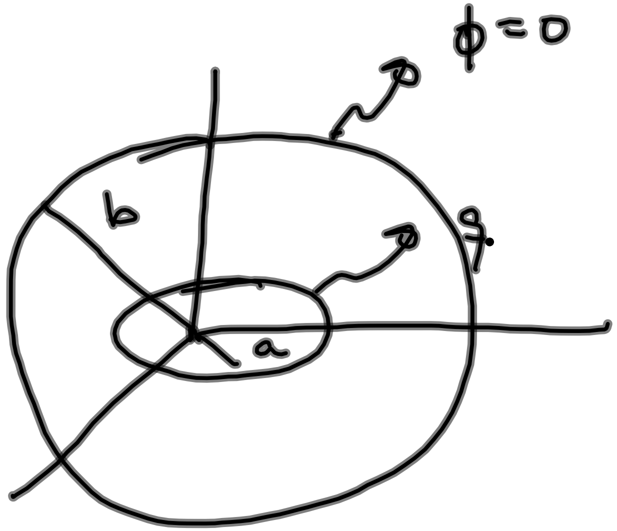
With the expansion, you can use $G(\bar{r},\bar{r}')$ to solve:



$$\rho(\bar{x}) = \frac{q}{2\pi a^2}\,\delta(r-a)\,\delta(\cos\theta)$$

Using G you can find $\phi(\bar{r})$ inside V:

$$\phi(\bar{r}) = \int_V \rho(\bar{x}')\,G(\bar{x},\bar{x}')\,d^3x'$$

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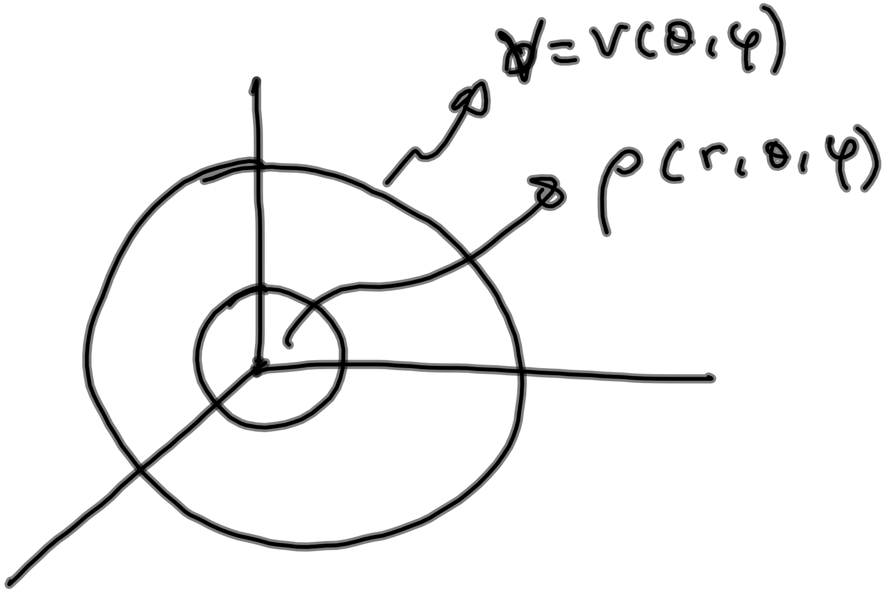
You also can do this:



you still can find $\phi(\bar{r})$ inside V even inside the region with charge.

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