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Tensors in Relativity

Ch. 4.5 (book). Uses SI system

We will use Gaussian units since there are fewer constants around.

The laws of physics have to be invariant under Lorentz transformations. Using tensors we can write equations in covariant form.

1) Invariance under space and time translations (space/time are homogeneous)

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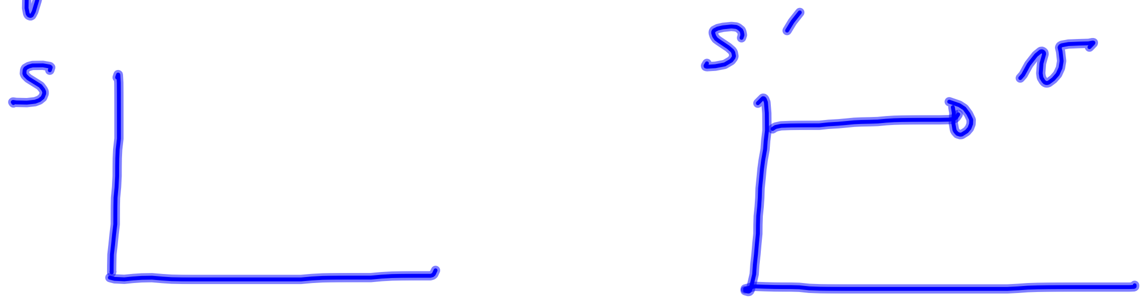
2) Invariance under rotations in 3 dimensions because space is isotropic.

3) Invariance under Lorentz transformations because Maxwell's equations satisfy this.

If



so that at $t = t' = 0$ $S \equiv S'$

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If we turn on an spherical wave of light at $t = t' = 0$ and $\bar{r} = \bar{r}' = 0$ then under a Lorentz transformation:

$$c^2t^2 - x_1^2 - x_2^2 - x_3^2 = c^2t'^2 - x_1'^2 - x_2'^2 - x_3'^2$$

Because c is the same in S and S' and the wave front at r or r' travels at the speed of light.

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Let's define: $x^0 = ct \quad x^1 = x \quad x^2 = y \quad x^3 = z$

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

$$\parallel$$

$$dr_\mu dr^\mu = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j \, dx^i dx^j = g_{ij} \, dx^i dx^j$$

Then:

$$g_{ij} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$g_{00} = 1$

$g_{11} = -1$

metric tensor.

$$g^{ij} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$ since $g^{ij} g_{jk} = \delta^i{}_k$

Let's define:

$$x^\mu = (\underbrace{x^0}_{\text{scalar in 3D.}}, \underbrace{\bar{x}}_{\text{3D vector}})$$ contravariant

$$x_\mu = (x^0, -\bar{x})$$ covariant

$$= g_{\mu\nu} x^\nu$$

Scalar product:

$$x_\mu x^\mu = g_{\mu\nu} x^\nu x^\mu = x^{0\,2} - |\bar{x}|^2$$

Derivatives:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \bar{\nabla})$$ covariant derivative.

then

$$\partial^\mu = g^{\mu\nu}\partial_\nu = (\partial_0, -\bar{\nabla})$$ contravariant derivative.

$$\partial^2 = \Box = \partial_\mu \partial^\mu = (\partial_0^2, -\nabla^2)$$ d'Alamberdian

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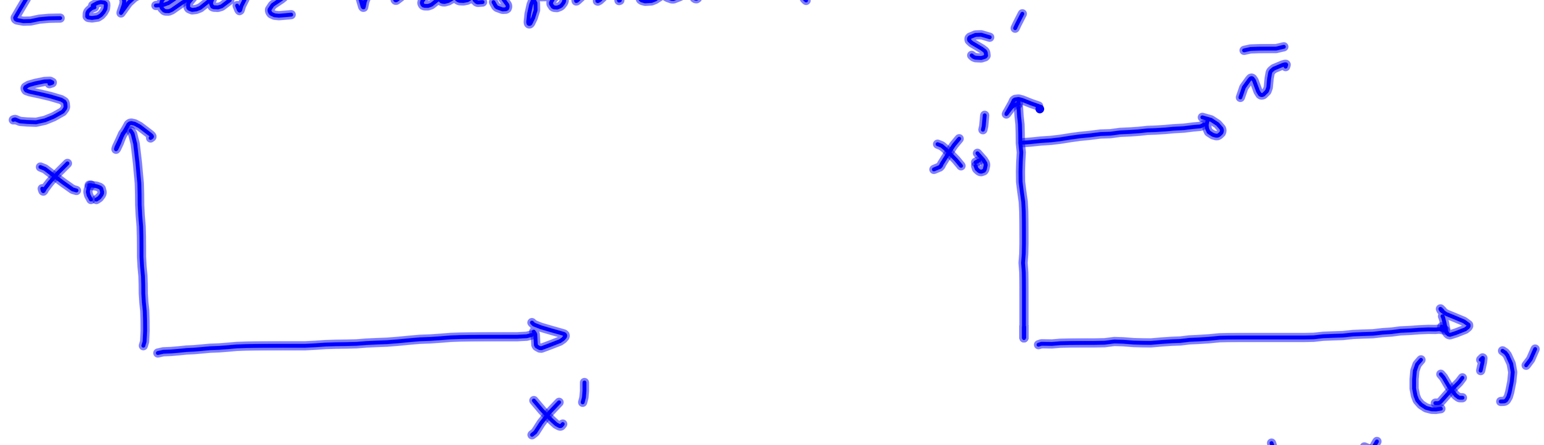
Lorentz transformation:



Using that for a spherical wave of light $ct = \bar{r}$ and $ct' = \bar{r}'$ one can get:

$$x'^0 = \gamma x^0 - \beta\gamma x^1$$
$$x'^1 = -\beta\gamma x^0 + \gamma x^1$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

$$\beta = \frac{v}{c} \leq 1$$

$$\gamma = (1-\beta^2)^{-1/2}$$

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$$M^i{}_j = \frac{\partial x'^i}{\partial x^j} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix}$$

Some people defines

$\gamma = \cosh \rho$

$\beta\gamma = \sinh \rho$

$\tanh \rho = \beta = v/c$

Lorentz transformation can be thought as a rotation by an imaginary angle $i\rho$.

Alternative way of defining the variables:

$x^1 = x \qquad x^2 = y \qquad x^3 = z \qquad x^4 = ict$

$$ds^2 = dx^{1\,2} + dx^{2\,2} + dx^{3\,2} + dx^{4\,2}$$

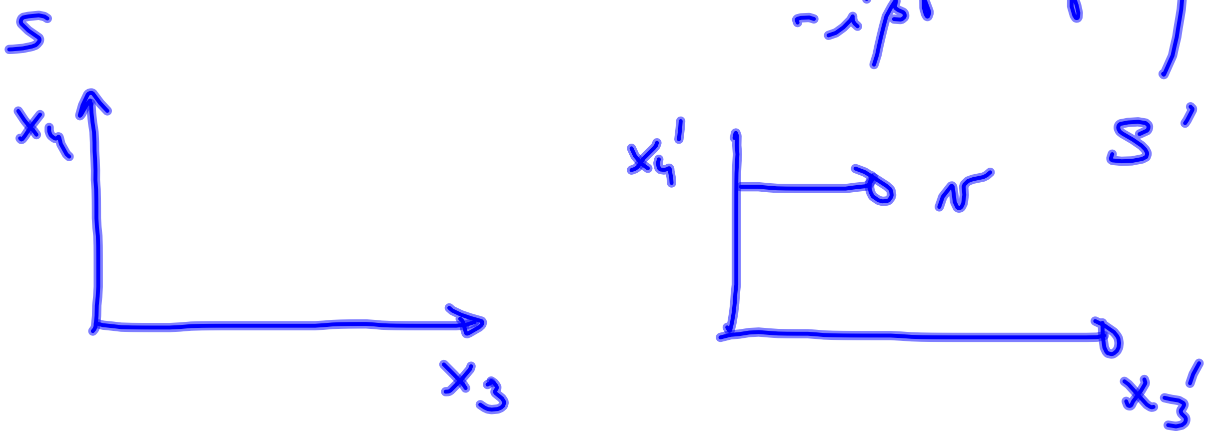
$$M^a{}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}$$

Here

$\gamma = \cos\theta$

$i\beta\gamma = \sin\theta$

$\tan\theta = i\frac{v}{c}$



Levi-Civita tensor in 4-D Minkowsky space:

$$\varepsilon^{ijkl} = \begin{cases} 1 & \text{if } ijkl \text{ are in cyclic order} \\ -1 & \text{"} \quad \text{"} \quad \text{are not in " "} . \\ 0 & \text{if there are repeated indices.} \end{cases}$$

$$\varepsilon_{ijkl} = g_{ir}\, g_{js}\, g_{kt}\, g_{lu}\, \varepsilon^{rstu}$$

if $rstu = 0123$ $\quad \varepsilon^{0123} = 1$

$$\varepsilon_{0123} = g_{00}\, g_{11}\, g_{22}\, g_{33}\, \varepsilon^{0123} = 1(-1)(-1)(-1)\,1 = -1$$

$$\varepsilon_{0123} = -\varepsilon^{0123}$$

Different than in 3D space.

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4- divergence:

$$\partial_\alpha A^\alpha = \partial_0 A^0 + \bar{\nabla}\cdot\bar{A} \qquad \text{scalar}$$

$(\partial_0, \bar{\nabla})$ $(A^0, \bar{A})$

Invariants :

consider :

$A^i = (dx^0, 0, 0, 0)$ $C^k = (0, 0, dx^2, 0)$

$B^j = (0, dx^1, 0, 0)$ $D^l = (0, 0, 0, dx^3)$

$$H^{ijkl} = A^i B^j C^k D^l$$

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Calculate:

$$\varepsilon_{ijkl} H^{ijkl} = dx^0 dx^1 dx^2 dx^3 = d^4x$$

volume element in 4D.

But notice that

$dx^1 dx^2 dx^3$ is not invariant.

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Continuity equation:

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot \underbrace{(\rho \bar{v})}_{\bar{J} \rightarrow \text{current}} = 0$$

If we define

$$J^\mu = (c\rho, \bar{J})$$ then ① becomes

$$\frac{c\,\partial \rho}{c\,\partial t} = \frac{\partial (c\rho)}{\partial (ct)} = \frac{\partial J^0}{\partial x^0} \qquad \bar{\nabla} \cdot \bar{J} = \partial_i J^i$$

with $i = 1, 2, 3$

$$\boxed{\partial_\mu J^\mu = 0}$$

In the book

$$J^\mu = \left(\rho, \frac{\rho \bar{v}}{c}\right) = \left(\rho, \frac{\bar{J}}{c}\right)$$

because of the different units.

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Another 4-vector is

$$A^\mu = (\varphi, \bar{A})$$

$\bar{A}$: vector potential

φ : scalar potential

$$\bar{B} = \bar{\nabla} \times \bar{A} \quad ②$$

$$\bar{E} = -\frac{1}{c}\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\varphi \quad ①$$

Book

$$A^\mu = \varepsilon_0 (\varphi, c\bar{A})$$

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\varphi$$

SI system

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Both $\bar{A}$ and φ satisfy wave equations:

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = 4\pi\rho \\ \dfrac{1}{c^2}\dfrac{\partial^2\bar{A}}{\partial t^2} - \nabla^2\bar{A} = \dfrac{4\pi}{c}\bar{J} \end{cases}$$

with $J^\nu = (c\rho, \bar{J})$ and $A^\nu = (\varphi, \bar{A})$

$$\boxed{\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu}$$

$$\frac{\partial^2}{c^2\partial t^2}\varphi$$

If $\nu = 0$

$$\partial_\mu \partial^\mu A^0 = \frac{4\pi}{c} J^0 \longrightarrow \frac{\partial^2}{(\partial x^0)^2}\varphi - \nabla^2\varphi = \frac{4\pi}{\cancel{c}}\cancel{c}\rho$$

Lorentz gauge:

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} + \bar{\nabla}\cdot\bar{A} = 0 \longrightarrow \partial_\alpha A^\alpha = 0$$

Now define:

$$F^{\mu\lambda} = \partial^\mu A^\lambda - \partial^\lambda A^\mu = -F^{\lambda\mu}$$

antisymmetric tensor of rank 2

$F^{\mu\lambda}$ can be expressed in terms of E_i and B_j (electric and magnetic fields).

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Calculate

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{\partial \varphi}{\partial x} = -E_x$$

from ①

then doing that for all elements:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Define $\mathcal{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$ dual tensor of $F_{\gamma\delta}$

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$$F_{\alpha\beta} = g_{\alpha\gamma}\, g_{\beta\delta}\, F^{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{01} = g_{0\gamma}\, g_{1\delta}\, F^{\gamma\delta} =$$

$$= g_{00}\, g_{11}\, F^{01} =$$

$$= -F^{01}$$

$$\mathcal{F}^{01} = \frac{1}{2}\, \varepsilon^{01\gamma\delta} F_{\gamma\delta} = \frac{1}{2}\, \overset{1}{\overset{\|}{\varepsilon^{0123}}}\, \overset{-B_x}{\overset{\|}{F_{23}}} +$$

$$+ \frac{1}{2}\, \overset{-1}{\overset{\|}{\varepsilon^{0132}}}\, \underset{B_x}{\underset{\|}{F_{32}}} =$$

$$= -B_x$$

Then

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

pseudotensor of rank 2.

Now

$$\left.\begin{aligned} \bar{\nabla}\cdot\bar{E} &= 4\pi\rho \\ \bar{\nabla}\times\bar{B} - \frac{1}{c}\frac{\partial\bar{E}}{\partial t} &= \frac{4\pi\bar{J}}{c} \end{aligned}\right\} \quad \partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

and

$$\left.\begin{aligned} &\bar{\nabla} \cdot \bar{B} = 0 \\ &\bar{\nabla} \times \bar{E} + \frac{1}{c}\frac{\partial \bar{B}}{\partial t} = 0 \end{aligned}\right\} \partial_\alpha \mathcal{F}^{\alpha\beta} = 0$$