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Maxwell eq's.

Last time we showed that

$$\left.\begin{aligned} \bar{\nabla}\cdot\bar{B} &= 0 \\ \bar{\nabla}\times\bar{E} + \frac{1}{c}\frac{\partial\bar{B}}{\partial t} &= 0 \end{aligned}\right\} \quad \partial_\alpha \mathcal{F}^{\alpha\beta} = 0$$

Also you can use $F_{\alpha\beta}$ (covariant stress tensor):

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

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Example:

If $\alpha = 1 \quad \beta = 2 \quad \gamma = 3$

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

$$\frac{\partial}{\partial x}(-B_x) + \frac{\partial}{\partial y}(-B_y) + \frac{\partial}{\partial z}(-B_z) = 0$$

$$\Rightarrow \bar{\nabla} \cdot \bar{B} = 0.$$

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Differential Equations

In Physics:

$\nabla^2 \psi = 0$ Laplace (homogeneous)

$\bar{F} = m \ddot{\bar{x}}$ Newton's 2nd law.

$\nabla^2 \psi = -\frac{\rho}{\varepsilon_0}$ Poisson (inhomogeneous)

$\nabla^2 \psi + k^2 \psi = 0$ wave equation or Helmholz equation.

$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = i\hbar \frac{\partial \psi}{\partial t}$ Schrödinger's eq.

We are going to learn a variety of techniques:

1) Separation of variables (for homogeneous eqs. but can be used for some non-homogeneous cases).

2) Green Functions (inhomogenous d.e.)

3) Frobenius method (certain ordinary diff. eqs.).

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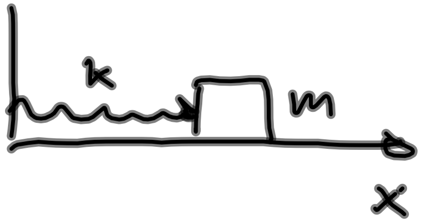
Boundary conditions: (BC):

BC's are necessary to determine the values of the coefficients of the general solution of the D.E.

The number of needed B.C.'s is equal to the order of the equation.

Example: harmonic oscillator in 1D:



$m\ddot{x} + kx = 0$ (1) $F = -kx$

$x = A\cos(\omega t + \varphi)$ (2)

with $\omega = \sqrt{\frac{k}{m}}$ (from plugging (2) in (1)).

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A and φ are determined by the B.C.
(initial conditions in this problem).

x_0 and v_0 are the initial conditions

$x_0 = x(t=0)$ $v_0 = \dot{x}(t=0)$

$x_0 = A\cos(\varphi)$ at $t=0$

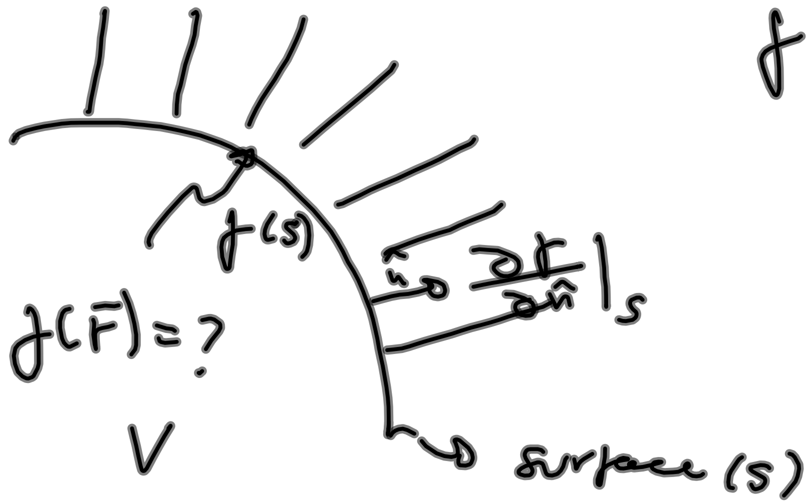
$\dot{x}(t=0) = -A\omega\sin(\varphi) = v_0$

$$\boxed{\tan\varphi = -\frac{v_0}{x_0\omega}} \quad \text{and} \quad \boxed{A = \frac{x_0}{\cos\varphi}}$$

Problems defined inside finite volumes:

1) Cauchy boundary conditions: $f(s)$ and

$\left.\frac{\partial f}{\partial \hat{n}}\right|_s$. f : function $\hat{n}$: normal



$f(\bar{r})$ solves the differential eq. inside V (defined by the surface S).

Dirichlet B.C : for problems in an enclosed volume (to solve Poisson's eq. for example).

you provide $f(s)$ - value of the function on the surface.

or

Neumann B.C : same problems as above (Poisson's eq. in enclosed volume).

you provide $\left.\frac{\partial f}{\partial \hat{n}}\right|_s$

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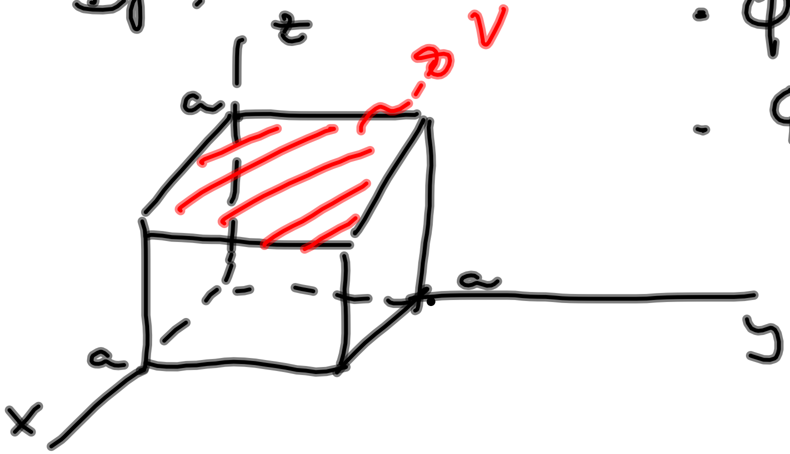
Separation of Variables

Example :

$$\nabla^2 \phi = 0 \quad (1)$$ Laplace equation.

The solution depends on the geometry of your problem that is defined by the B.C.

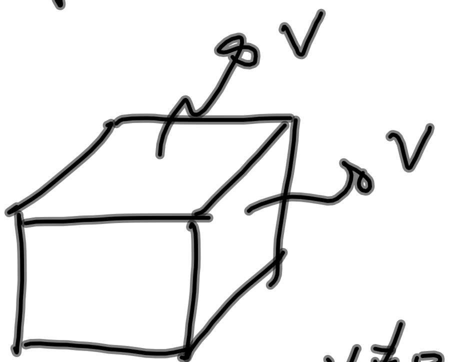
If :



- $\phi = V$ for $z = a$
- $\phi = 0$ on all other surfaces
- Find ϕ inside the cube.

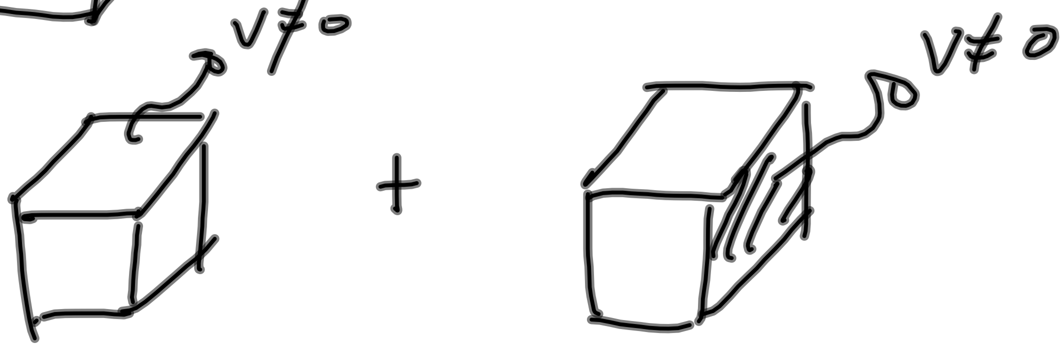
Many times you need to use the principle of superposition!

If the problem is this!



$\Phi = 0$ in all faces but 2.

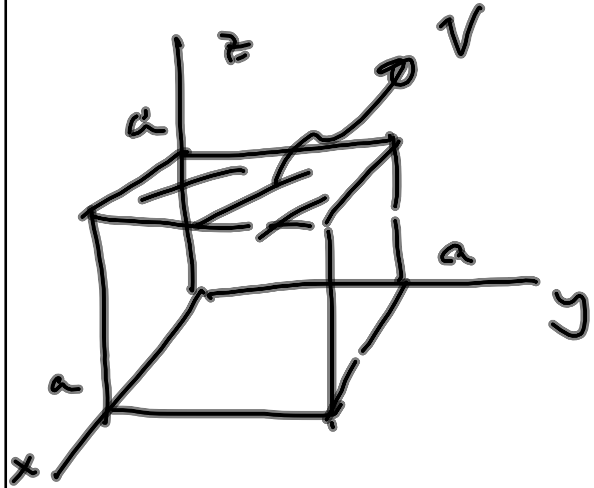
Then you need to add the solutions for:



Then:

Assume that the solution has the form:

(2) $\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$



this is a very common situation in physics.

Plugg (2) in (1):

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

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$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} = 0 \quad (3)$$

Divide (3) by $\phi = XYZ$:

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-\beta^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{\gamma^2 = \alpha^2+\beta^2} = 0$$

Since $x, y,$ and z can vary independently each term has to be equal to a constant.

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Now I got 3 ordinary diff. eqs. .

$$\frac{\partial^2 X}{\partial x^2} = \alpha^2 X \Rightarrow X(x) \propto \begin{cases} \cos \alpha x \\ \sin \alpha x \end{cases}$$ solutions

$$\frac{\partial^2 Y}{\partial y^2} = \beta^2 Y \Rightarrow Y(y) \propto \begin{cases} \cos \beta y \\ \sin \beta y \end{cases}$$

$$\frac{\partial^2 Z}{\partial z} = -\gamma^2 Z \Rightarrow Z(z) \propto \begin{cases} e^{\gamma z} \\ e^{-\gamma z} \end{cases} \text{ or } \begin{matrix} \cosh \gamma z \\ \sinh \gamma z \end{matrix}$$

General solution :

$$\phi(x,y,z) = \sum_{\alpha,\beta} (A_\alpha \cos \alpha x + B_\alpha \sin \alpha x)(A_\beta \cos \beta y + B_\beta \sin \beta y)(A_{\alpha\beta} \cosh \gamma z + B_{\alpha\beta} \sinh \gamma z)$$

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In our problem we know that

$$\phi(x=0,\ y=0,\ z=0)=0$$

Since $\phi(x=0) = \phi(y=0) = \phi(z=0) = 0$

then $A_\alpha = A_\beta = A_{\alpha,\beta} = 0$

Now we have

$$\phi(x,y,z) = \sum_{\alpha,\beta} \overbrace{B_\alpha B_\beta B_{\alpha\beta}} \sin\alpha x \sin\beta y \sinh \gamma z.$$

Since

$$\phi(x=a, y, z) = 0 \Rightarrow \sin\alpha a = 0 \Rightarrow \alpha = \frac{\pi n}{a} \equiv \alpha_n$$

$n = 1, 2, 3, \ldots$

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Since

$$\phi(x, y=a, z) = 0 \Rightarrow \sin \beta a = 0 \Rightarrow \beta = \frac{\pi m}{a}$$

$m = 1, 2, \ldots$

Then we have that

$$\phi(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{n,m} \sin \frac{\pi n x}{a}$$

$$\sin \frac{\pi m y}{a} \sinh \gamma_{nm} z$$

$$\gamma_{nm} = \left[\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{a}\right)^2 \right]^{1/2} = \frac{\pi}{a} \sqrt{n^2 + m^2}$$

$A_{n,m}$ is found using the remaining B.C.

$\phi(x, y, z = a) = V$ or any $f(x, y)$

We will use that the $\sin\frac{m\pi y}{a}$ and $\sin\frac{n\pi x}{a}$ are a set of orthogonal functions in the intervals $(0, a)$ for x and y.

So any function $f(x,y)$ can be expanded in terms of these.

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Now

$$\phi(x,y,z=a) = \underbrace{\phi_0(x,y)}_{V \text{ in our case}} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{a}$$

$$\sin\frac{m\pi y}{a} \sinh\left\{\frac{\pi}{\cancel{a}}(n^2+m^2)^{1/2}\cancel{a}\right\}$$

Multiply both sides by $\sin\frac{n'\pi x}{a} \sin\frac{m'\pi y}{a}$ and integrate over x and y in the interval $(0,a)$ and use that:

$$\int_0^a \sin\frac{n x\pi}{a} \sin\frac{n' x\pi}{a}\, dx = \frac{a}{2}\delta_{n,n'} \quad \text{(orthogonality condition)}$$

$$\int_0^a dx \int_0^y dy \; \phi_0(x,y) \sin\frac{n'\pi x}{a} \sin\frac{m'\pi y}{a} =$$

$$= \sum_{n,m=1}^{\infty} A_{nm} \sinh\left[\pi (n^2+m^2)^{1/2}\right] \underbrace{\int_0^a dx \sin\frac{n'\pi x}{a} \sin\frac{n\pi x}{a}}_{\frac{a}{2}\delta_{n,n'}}$$

$$\underbrace{\int_0^a dy \sin\frac{m'\pi y}{a} \sin\frac{m\pi y}{a}}_{\delta_{m,m'}\frac{a}{2}} =$$

$$= \frac{a^2}{4} A_{n'm'} \sinh\left[\pi (n^2+m^2)^{1/2}\right]$$

Then:

$$\boxed{A_{nm} = \frac{4}{a^2}\,\frac{1}{\sinh\left[\pi\,(n^2+m^2)^{1/2}\right]}\int_0^a dx\int_0^a dy\;\phi_0(x,y)\,\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{a}}$$

For $\phi_0(x,y) = V$

$$A_{nm} = \frac{4}{a^2}\,\frac{V}{\sinh\left[\pi\,(n^2+m^2)^{1/2}\right]}\underbrace{\int_0^a dx\,\sin\frac{n\pi x}{a}}_{\underbrace{-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\Big|_0^a}_{-\frac{a}{n\pi}\left[(-1)^n-1\right]}}\;\underbrace{\int_0^a dy\,\sin\frac{m\pi y}{a}}_{-\frac{a}{m\pi}\left[(-1)^m-1\right]}$$

We see that only odd values of n and m survive so define $n = 2k+1$

$m = 2j+1$

and we get that since

$$A_{nm} = \frac{16}{nm\pi^2} \frac{V}{\sinh\left[\pi (n^2+m^2)^{1/2}\right]}$$

$$A_{jk} = \frac{16}{(2k+1)(2j+1)\pi^2} \frac{V}{\sinh\left[\pi\left[(2j+1)^2+(2k+1)^2\right]^{1/2}\right]}$$

Then:

$$\phi(x,y,z) = \frac{16V}{\pi^2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sin\frac{(2k+1)\pi x}{a} \sin\frac{(2j+1)\pi y}{a}$$

$$\sinh\left\{\frac{\pi}{a}\left[(2k+1)^2+(2j+1)^2\right]^{1/2} z\right\}\left[\sinh\left\{\pi\left[(2k+1)^2+(2j+1)^2\right]^{1/2}\right\}\right]^{-1}$$

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