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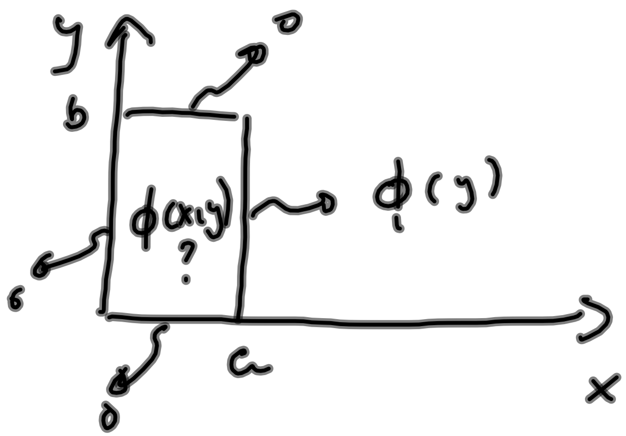
Separation of Variables

cartesian coordinates

Example in 2 dimensions.



$\phi(x,y) = ?$

$\nabla^2 \phi(x,y) = 0$

inside the "volume".

$\phi(x,y) = X(x)\, Y(y)$ assumption.

Then

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{dx^2}}_{+\alpha^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-\alpha^2} = 0$$

periodic solutions along y because in our problem

$\phi(x, y=0) = \phi(x, y=b) = 0.$

Then

$X(x) \propto e^{\pm\alpha x} \propto \cosh \alpha x$ and $\sinh \alpha x$

$Y(y) \propto e^{\pm i\alpha y} \propto \sin \alpha y$ and $\cos \alpha y$

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Since $\phi(x,0)=0$ $\quad \phi(x,y) \propto \sin ky$

Since $\phi(x,b)=0$ $\quad \sin kb = 0 \Rightarrow k = \frac{\pi n}{b}$

since $\phi(0,y)=0$ $\quad \phi(x,y) \propto \sinh kx$ $\quad n=1,2,\ldots$

Then

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \sinh \frac{\pi n x}{b} \sin \frac{\pi n y}{b}$$

To find A_n we use that $\phi(a,y) = \phi(y)$

$$\phi(a,y) = \phi(y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b}$$

Multiply both sides by $\sin \frac{n'\pi y}{b}$ and integrate over y between 0 and b:

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$$\int_0^b \phi(y) \sin\frac{n'\pi y}{b}\, dy = \sum_{n=1}^{\infty} A_n \sinh\frac{\pi n a}{b} \underbrace{\int_0^b \sin\frac{n\pi y}{b} \sin\frac{n'\pi y}{b}\, dy}_{\frac{b}{2}\delta_{n,n'}}$$

If $\phi(y) = V$

$$V \underbrace{\int_0^b \sin\frac{n'\pi y}{b}\, dy}_{-\frac{b}{n'\pi}\cos\frac{n'\pi y}{b}\Big|_0^b} = A_{n'} \frac{b}{2} \sinh\frac{\pi n' a}{b}$$

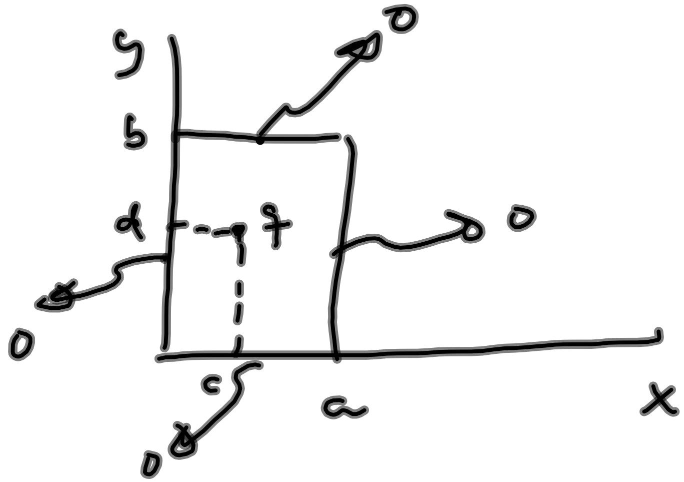
$$-\frac{b}{n'\pi}\cos\frac{n'\pi y}{b}\Big|_0^b = -\frac{b}{n'\pi}\underbrace{(\underbrace{\cos n'\pi}_{(-1)^{n'}} - 1)}_{\substack{0 \text{ for } n' \text{ even} \\ -2 \text{ for } n' \text{ odd}}}$$

$$A_n = \frac{4V}{n\pi \sinh\frac{\pi n a}{b}} \text{ for } n \text{ odd} \qquad A_n = 0 \text{ for } n \text{ even.}$$

Then defining $n = 2j+1$

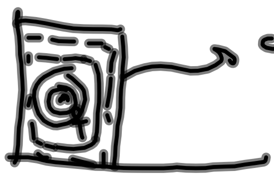
$$\boxed{\phi(x,y) = \frac{4V}{\pi} \sum_{j=0}^{\infty} \frac{\sinh\frac{(2j+1)\pi x}{b}\,\sin\frac{(2j+1)\pi y}{b}}{\sinh\frac{(2j+1)\pi a}{b}}}$$

Now consider this problem:

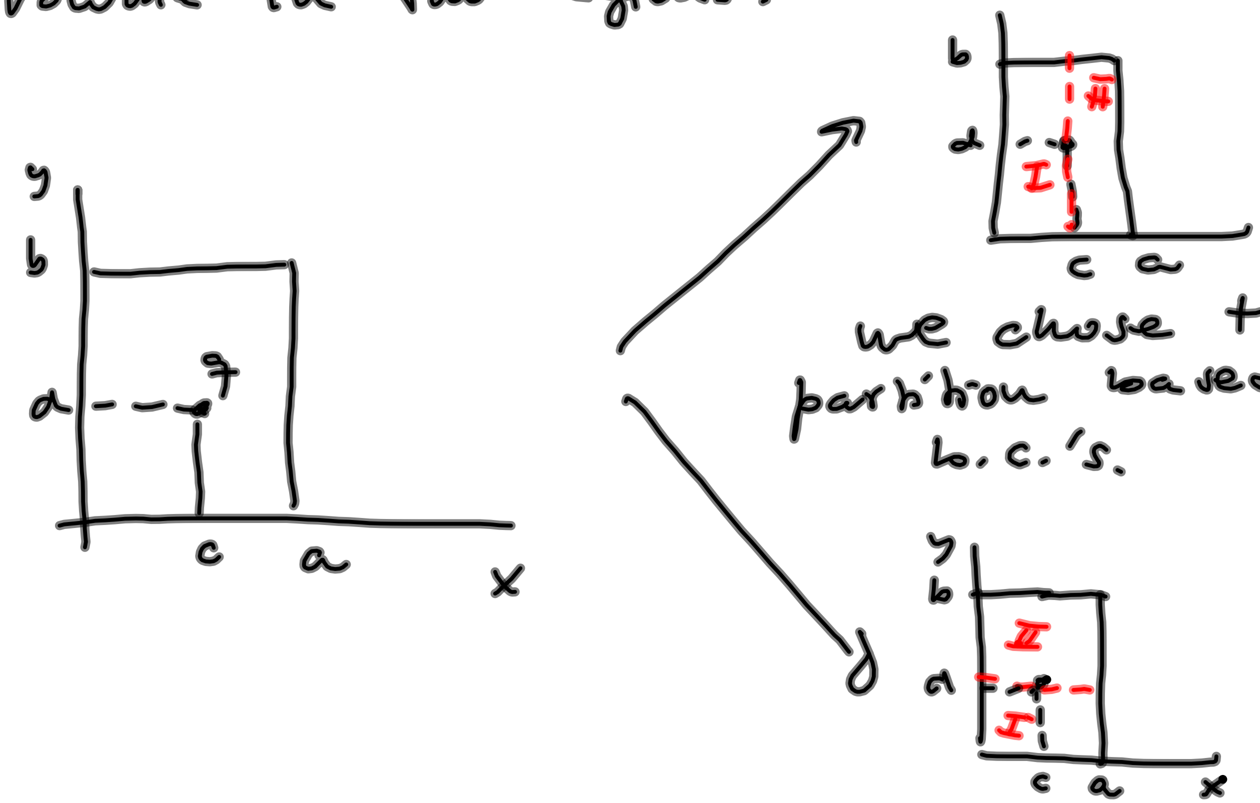


In space

$$\phi_q = \frac{q}{4\pi\varepsilon_0 r}$$



Since we can use sep. of variables to solve $\nabla^2 \phi = 0$ we need to separate our volume in two regions.



we chose the partition based on the b.c.'s.

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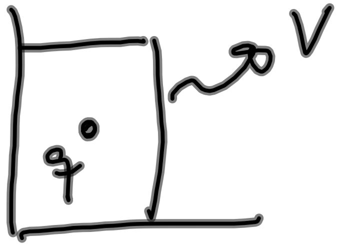
Since I plan to combine the previous problem with this one:



$\phi(x,y) = ?$

we are going to chose position I

Region I : $0 \le x \le c$

Region II : $c \le x \le a$

Then we do the following -

we propose a solution for each region:

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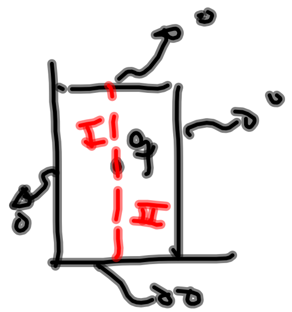
$$\phi^{I}(x,y) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi x}{b} \sin\frac{n\pi y}{b}$$



$$\phi^{II}(x,y) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi (a-x)}{b} \sin\frac{n\pi y}{b}$$

To obtain A_n and B_n we need two b.c. at $x=c$.

i) $\phi^{I}(c,y) = \phi^{II}(c,y)$ because $\phi(x,y)$ has to be continuous.

$$A_n \sinh\frac{n\pi c}{b} = B_n \sinh\frac{n\pi(a-c)}{b} \qquad (1)$$

$$A_n = B_n \sinh\frac{n\pi(a-c)}{b} \Big/ \sinh\frac{n\pi c}{b}$$

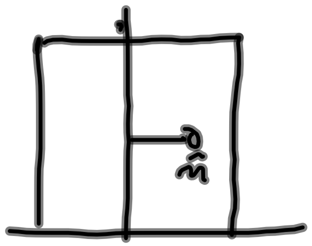
We know that the normal component of the electric field has a jump proportional to the density of charge on the surface:

charge on the surface

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \Rightarrow E_n^{II} - E_n^{I} = \frac{\rho}{\varepsilon_0} = \frac{q\,\delta(y-d)}{\varepsilon_0}$$

$$E_n = -\frac{\partial \phi}{\partial \hat{n}} = -\frac{\partial \phi}{\partial x}$$

$$-\frac{\partial \phi^{II}}{\partial x} + \frac{\partial \phi^{I}}{\partial x} = \frac{q\,\delta(y-d)}{\varepsilon_0}$$



C

$\hat{n} = x$

$\hat{n}$ has always to be the normal outside the volume.

$$\sum_{n=1}^{\infty} B_n \frac{n\pi}{b} \cosh \frac{n\pi(a-c)}{b} \sin \frac{n\pi y}{b} +$$

$$+ \sum_{n=1}^{\infty} A_n \frac{n\pi}{b} \cosh \frac{n\pi c}{b} \sin \frac{n\pi y}{b} = \frac{q}{\varepsilon_0} \delta(y-d)$$

Using ① and combining terms:

$$\sum_{n=1}^{\infty} B_n \frac{n\pi}{b} \sin \frac{n\pi}{b} y \left[\cosh \frac{n\pi(a-c)}{b} + \cosh \frac{n\pi c}{b} \times \right.$$

$$\left. \times \frac{\sinh \frac{n\pi(a-c)}{b}}{\sinh \frac{n\pi c}{b}} \right] = \frac{q}{\varepsilon_0} \delta(y-d) \qquad ②$$

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Multiply ② by $\sin\frac{n\pi y}{b}$ and integrate over y.

Using orthogonality we get:

$$B_n \frac{n\pi}{b}\frac{b}{2}\left[\cosh\frac{n\pi(a-c)}{b} + \cosh\frac{n\pi c}{b}\frac{\sinh n\pi(a-c)/b}{\sinh\frac{n\pi c}{b}}\right]$$

$$= \frac{q}{\varepsilon_0}\sin\frac{n\pi d}{b}$$

From this we get B_n as a function of the geometry of the problem a, b, c, d.

We find A_n plugging B_n in ① and replacing A_n and B_n in ϕ^{I} and ϕ^{II} we get:

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$$\phi_I(x,y) = \frac{2q}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi d}{b}\,\frac{\sinh \pi n\frac{(a-c)}{b}}{\sinh\frac{\pi n a}{b}}$$

$$\times \sin\frac{n\pi y}{b}\sinh\frac{\pi n x}{b} \qquad \text{for } x \leqslant c$$

$$\phi_{II}(x,y) = \frac{2q}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi d}{b}\,\frac{\sinh\frac{\pi n c}{b}}{\sinh\frac{\pi n a}{b}}\times$$

$$\times \sin\frac{n\pi y}{b}\sinh\frac{\pi(a-x)n}{b}$$

$$\text{for } x \geqslant c.$$

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This can be written in a compact way:

$$\phi(x,y) = \frac{2q}{\pi \varepsilon_0} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi a}{b} \sin\frac{n\pi y}{b} \times$$

$$\times \frac{\sinh\frac{\pi n x_<}{b} \sinh\frac{\pi n (a - x_>)}{b}}{\sinh\frac{\pi n a}{b}}$$

where $x_<$ $(x_>)$ is the smaller (larger) between c and x.

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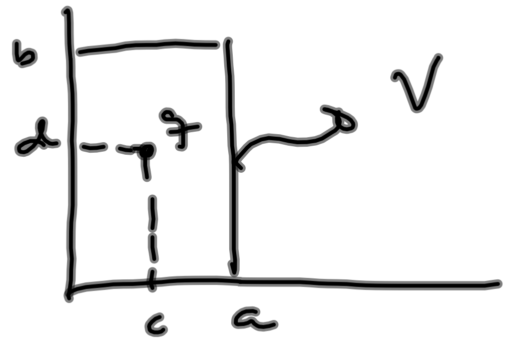
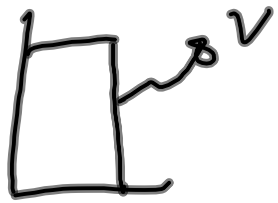
Now if you want to find $\phi(x,y)$ for



all you have to do is to add the solution to



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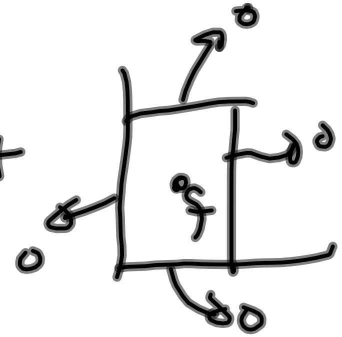

using the principle of superposition.

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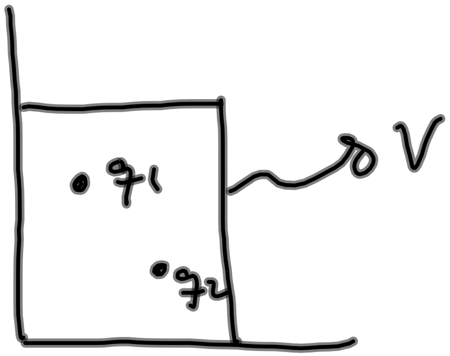
You also could solve the problem the "hard way":

$$\phi^{I}(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi x}{b}$$

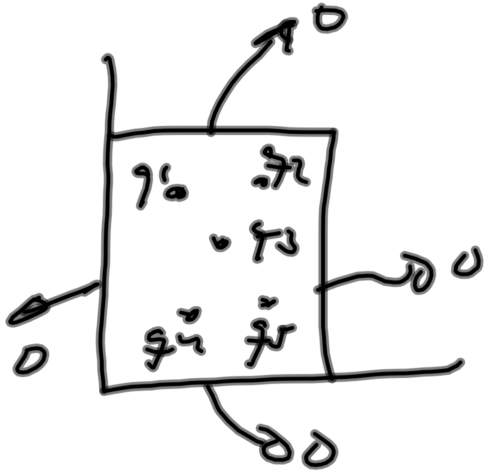
$$\phi^{II}(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b}\left[B_n e^{\frac{n\pi x}{b}} + C_n e^{-\frac{n\pi x}{b}}\right]$$

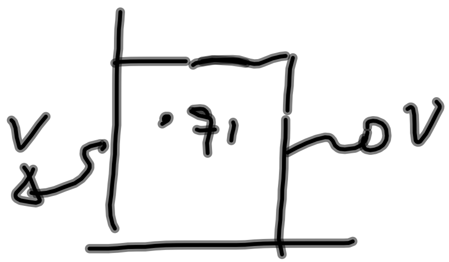
You solve for A_n, B_n and C_n using 2 b.c.'s at $x=c$ and 1 b.c. at $x=a$.

Notice that now using superposition you can solve problems like this:



or





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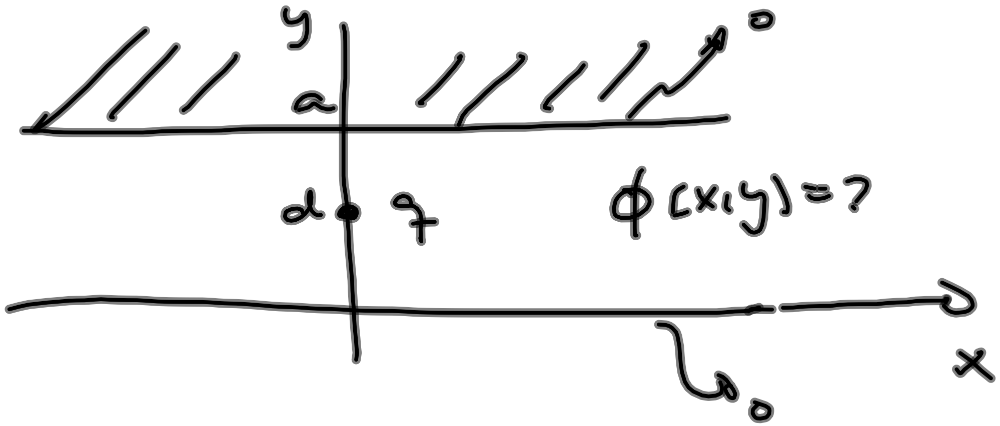
Hw:



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