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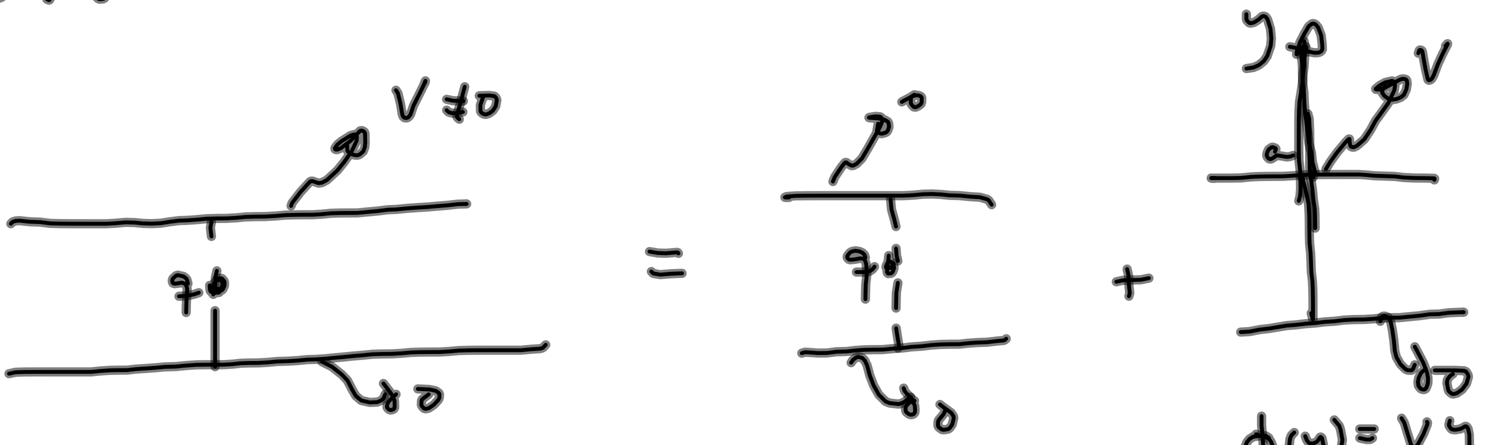
Hw #8 - Pr 6:



$\phi(y) = \frac{Vy}{a}$

"trivial" solution

but what about using the solutions we found for $\nabla^2\phi(x,y)=0$ in cartesian coordinates?

$$\nabla^2\phi = \nabla^2(X(x)\,Y(y)) = Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

But here $X(x)=1$ (since no x dependence)

If $X(x) = 1$ we get

$$\nabla^2 \phi = 0 \Rightarrow \frac{\partial^2 Y}{\partial y^2} = 0$$ (instead of $\pm k^2 \neq 0$!)

Then $Y(y) \propto y$.

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Laplace Eq. in Spherical coordinates

$$\nabla^2 \phi = 0 \qquad \phi = \phi(r,\theta,\varphi) = \frac{U(r)}{r} P(\theta) Q(\varphi) \quad (1)$$

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \phi}{\partial\theta}\right) +$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0 \quad (2)$$

Plugg (1) in (2) and multiply both sides by $r^2 \sin^2\theta / UPQ$:

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$$\underbrace{r^2 \sin^2\theta \left[\frac{1}{U}\frac{\partial^2 U}{\partial r^2} + \frac{1}{P r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{dP}{d\theta}\right)\right]}_{m^2} +$$

$$+ \underbrace{\frac{1}{Q}\frac{d^2 Q}{d\varphi^2}}_{-m^2} = 0$$

$$\therefore \quad Q(\varphi) \propto e^{\pm i m \varphi} \qquad m = 0, 1, 2, \ldots$$

If there is azimuthal symmetry so that

$\phi(\bar{r}) = \phi(r, \theta)$ then $Q(\varphi) = 1 \Rightarrow \boxed{m = 0}$.

Now we try to separate the part of the eq. equal to m^2:

$$\frac{r^2\sin^2\theta}{U}\frac{d^2U}{dr^2} + \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = m^2$$

÷ by $\sin^2\theta$:

$$\underbrace{\frac{r^2}{U}\frac{d^2U}{dr^2}}_{l(l+1)} + \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = \frac{m^2}{\sin^2\theta}$$

for convenience.

$$\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -l(l+1) \quad (1)$$

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We see that

$$\frac{d^2U}{dr^2} - \frac{\ell(\ell+1)}{r^2}U = 0$$ can be solved by

$$\boxed{U = A_\ell r^{\ell+1} + \frac{B_\ell}{r^\ell}}$$

Now we need to find $P(\theta)$ from Eq. (1):

Define $x = \cos\theta$ $\quad \sin\theta = (1-x^2)^{1/2}$

$dx = -\sin\theta\, d\theta$

Eq (1) becomes:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P(x) = 0 \qquad (2)$$

Generalized Legendre Equation

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Let's solve (2) for the case in which $m=0$, i.e., for problems with azimuthal symmetry:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0 \qquad (3)$$

for $-1 \le x \le 1$
since $x = \cos\theta$
$0 \le \theta \le \pi$

Let's use Frobenius method to solve Eq. (3):

$$P(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \qquad (4)$$

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$$\frac{d^2P}{dx^2} - x^2\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \ell(\ell+1)P = 0 \qquad (5)$$

Let's plug (4) in (5) and find a_λ and k;

$$\sum_{\lambda=0}^{\infty}\Big\{(k+\lambda)(k+\lambda-1)\,a_\lambda x^{k+\lambda-2} - \Big[\underbrace{(k+\lambda)(k+\lambda+1)}_{\text{from } (k+\lambda)(k+\lambda-1)+2(k+\lambda)} - \ell(\ell+1)\Big]a_\lambda x^{k+\lambda}\Big\} = 0$$

If $\lambda = 0$ we need to make 0 the coefficient for the lowest power of x which is x^{k-2}:

Then
$$k(k-1)a_0 = 0$$

If $a_0 \neq 0$ then $k=0$ or $k=1$.

If $\lambda = 1$ if $a_1 \neq 0$ we can make the lowest power of x vanish if $(k+1)k = 0$ then $k=0$ or $k=-1$ are solutions.

You can chose $a_0 \neq 0$ $a_1 = 0$ and $k=0, k=1$ are going to give you two solutions. One with even powers of x and the other with odd powers of x.

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The two same solutions will be obtained by choosing $a_0 = 0$ and $a_1 \neq 0$ with $k = 0$ and $k = -1$.

If we chose $a_0 \neq 0$ in order to make the coefficient of x^k vanish we need that

$$(k+\lambda+2)(k+\lambda+2-1)\, a_{\lambda+2} = \left[(k+\lambda)(k+\lambda-1) - \ell(\ell+1)\right] \times a_\lambda$$

then

$$a_{\lambda+2} = \frac{\left[(k+\lambda)(k+\lambda+1) - \ell(\ell+1)\right]}{(k+\lambda+1)(k+\lambda+2)}\, a_\lambda \qquad (6)$$

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If $a_0 \neq 0$ and $k=0$ then

$$P(x) = \sum_{j=0}^{\infty} a_{2j}\, x^{2j}$$

if $k=1$

$$P(x) = \sum_{j=0}^{\infty} a_{2j}\, x^{2j+1}$$

But $P(x=1)$ cannot diverge so the only way in which it won't diverge is if $a_{2j}=0$ for some values of $j \geq j_0$.

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Consider $\ell = 0$ and $k = 1$.

$$a_{\lambda+2} = \frac{(1+\lambda)(\lambda+2)}{(\lambda+2)(\lambda+3)} a_\lambda = \frac{(1+\lambda)}{(\lambda+3)} a_\lambda$$

$\lambda > 0$ then $a_\lambda \neq 0$ for all λ
and this solution is going to diverge at $x=1$.

But if $k=0$ we get:

$$a_{\lambda+2} = \frac{[\lambda(\lambda+1)]\, a_\lambda}{(\lambda+1)(\lambda+2)} = \frac{\lambda}{\lambda+2} a_\lambda$$

then if $\lambda = 0$ $a_2 = 0$ and then a_{2j} for $j \neq 0$ is zero.
Then $\boxed{P_{\ell=0}(x) = a_0}$

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So in general you will find that for

$\ell = 1$ the solution is $a_0 x$

$\ell = 2$ the solution is $a_0 + a_2 x^2$

$\ell = 3$ " " is $a_0 x + a_2 x^3$

etc.

$P_\ell(x)$ are polynomials of degree ℓ.

If ℓ is even only even powers of x appear.

If ℓ is odd only odd " " " ".

$P_\ell(x)$ are normalized so that $P_\ell(x=1) = 1$.

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Legendre Polynomials

Normalized they are:

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$\vdots$

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Properties of $P_\ell(x)$:

Rodrigues' formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell$$

$P_\ell(x)$ are orthogonal in the interval $[-1,1]$:

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1} \delta_{\ell,\ell'}$$

then

$$\int_{-1}^{1} P_\ell(x) \overset{P_0(x)}{1}\, dx = 2\,\delta_{\ell,0}$$

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However:

$$\int_0^1 P_\ell(x)\,dx = ?$$

Careful since $P_\ell(x)$ are not orthogonal in $[0,1]$.

Any well behaved function $f(x)$ in $[-1,1]$ can be expanded in terms of $P_\ell(x)$:

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)$$

$$\int_{-1}^{1} f(x) P_{\ell'}(x)\,dx = \sum_{\ell=0}^{\infty} A_\ell \underbrace{\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\,dx}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}} =$$

$$= \frac{2}{2\ell'+1} A_{\ell'}$$

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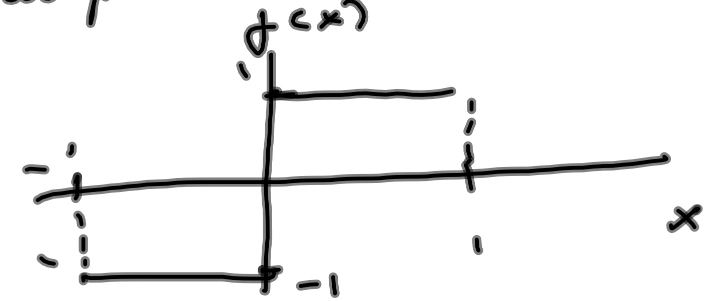
Then

$$\boxed{A_{\ell'} = \frac{2\ell'+1}{2}\int_{-1}^{1} f(x)\, P_{\ell'}(x)\, dx}$$

Example :



$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ -1 & \text{for } -1 \le x < 0 \end{cases}$$

$$A_\ell = \frac{2\ell+1}{2}\int_{-1}^{0} (-1)\, P_\ell(x)\, dx + \frac{2\ell+1}{2}\int_{0}^{1} P_\ell(x)\, dx$$

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If ℓ is even then $A_\ell = 0$ since $P_\ell(x) = P_\ell(-x)$.
If ℓ is odd $P_\ell(-x) = -P_\ell(x)$ and

$$A_\ell = (2\ell+1)\int_0^1 P_\ell(x)\,dx = \frac{\left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}}(2\ell+1)(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

$m!! = m(m-2)(m-4)\ldots$

for $\ell = 0$ do the $\int$.

General solution for
$\nabla^2\phi = 0$ with $\phi = \phi(r,\theta)$

$$\boxed{\phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)}$$

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A_e and B_e are determined by the b.c.'s.

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