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Further clasification of tensors.

Consider how a vector transforms upon the inversion transformation in 3D:

$S : \{x^i\}$ and $S' : \{x'^i\}$ $\quad x'^i = -x^i$

$$M^i{}_j = \frac{\partial x'^i}{\partial x^j} = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} \qquad \det|M| = -1$$

Prototype vector: r^i

$r'^i = -r^i$ in component

$\bar{r} = (x, y, z) = (-x', -y', -z') = \bar{r}'$

We see that $\bar{r}$ is <u>odd</u> under inversion because its components change sign.

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Now consider:

$$\bar{C} = \bar{A} \times \bar{B} \qquad \bar{A}, \bar{B} \text{ vectors.}$$

In components:

$$C^i = A^j B^k - A^k B^j$$

If we change to S' with $x'^i = -x^i$ we see that $A'^i = -A^i \quad B'^i = -B^i$

but then $C'^i = C^i$! The components do not change.

C^i is not a vector. It is a pseudovector.

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Vectors are also called polar vectors.

Pseudovector are called axial vectors.

We see that C transforms as:

$$c'^{i} = |\det M| \, M^{i}{}_{j} \, c^{j}$$

if $|\det M| = 1$ as in a rotation then C transforms like any vector

if $|\det M| = -1$ (inversion, reflexions) then C does not transform as a vector.

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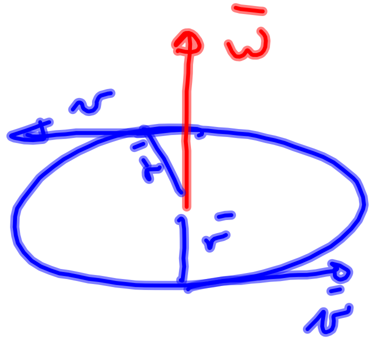
There are many axial vectors in physics:



$$\bar{\omega} = \frac{\bar{r} \times \bar{v}}{r^2}$$

$\bar{\omega}$ is a pseudovector

$\bar{r}, \bar{v}$ are vectors.

Other examples:

$\bar{L} = \bar{r} \times \bar{p}$ pseudovector.

But:

$\bar{v} = \bar{\omega} \times \bar{r}$ → polar

polar axial

Let's write the cross product in tensor notation:

$$L_i = \varepsilon_{ijk} r^j p^k$$

How does ε_{ijk} transform?

In 3D $\varepsilon_{ijk} = \varepsilon^{ijk}$

$$\varepsilon'^{ijk} = \frac{\partial x'^i}{\partial x^r} \frac{\partial x'^j}{\partial x^s} \frac{\partial x'^k}{\partial x^t} \varepsilon^{rst} =$$

$$= M^i{}_r M^j{}_s M^k{}_t \varepsilon^{rst} =$$

seen in previous lecture

$$= \det M \; \varepsilon^{ijk}$$

If M is a rotation $\det M = 1$ showing that ε^{ijk} is isotropic.

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isotropic means that the tensor has the same components in all systems of coordinates.

If $\det M = -1$ as in an inversion then

$$\varepsilon^{ijk\,\prime} = -\varepsilon^{ijk} = \det M \; \varepsilon^{ijk}$$

Since the $\det M$ appears in the transformation rule for ε^{ijk} then it is a pseudotensor.

$$\varepsilon_{ijk} = \bar{e}_i \cdot \bar{e}_j \times \bar{e}_k$$

$\hat{e}_i$: are vectors

$\hat{e}_j \times \hat{e}_k$: pseudovector

$\hat{e}_i \cdot (\bar{e}_j \times \bar{e}_k)$: pseudo tensor

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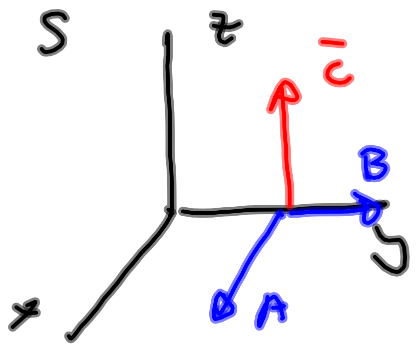
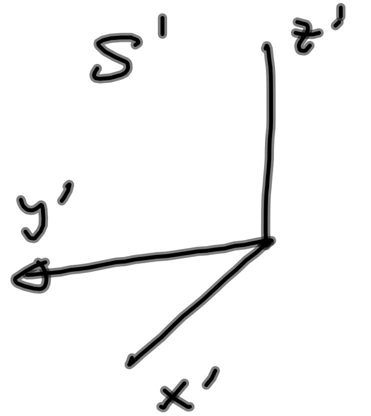
Now consider reflexions in 3D:





$x' = x$

$y' = -y$

$z' = z$

$$M^i{}_j = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix} \qquad \det M = -1$$

$$\bar{C} = \bar{A} \times \bar{B}$$

We see that $\bar{A}'$ and $\bar{B}'$ look like reflected on the x–z plane but $\bar{C}'$ seems to point in the opposite direction.

In general pseudo tensors include det M in their transformation. If det M = -1 then they transform differently than the same rank tensor.

Pseudoscalar:

$$S' = \det M \; S$$

Scalar

$$S' = S$$

Example:

$$S = \bar{A} \times \bar{B} \cdot \bar{C}$$

if $\bar{A}, \bar{B}$ and $\bar{C}$ are polar vectors then

$S' = -S$ upon an inversion.

S is a pseudoscalar.

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Notice:

scalar

$$S' = S$$

pseudoscalar

$$S' = \det M \, S$$

vector:

$$C'_i = \frac{\partial x^j}{\partial x'^i} C_j$$

pseudovector:

$$C'_i = \det M \frac{\partial x^j}{\partial x'^i} C_j$$

Tensor of rank 2:

$$A'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} A_{kl}$$

Pseudotensor of rank 2:

$$A'_{ij} = \det M \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} A_{kl}$$

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Properties of the cross-product (in 3D)

$\bar{C} = \bar{A} \times \bar{B}$ $\quad \bar{A}, \bar{B}$ are vectors.

Let's see that $\bar{C}$ is a pseudovector:

$$C^i = \varepsilon^{ijk} A_j B_k$$

$$C'^i = \varepsilon'^{ijk} A'_j B'_k = \det M \frac{\partial x'^i}{\partial x^r} \frac{\partial x'^j}{\partial x^s} \frac{\partial x'^k}{\partial x^t} \varepsilon^{rst}$$

$$\frac{\partial x^m}{\partial x'^j} A_m \frac{\partial x^n}{\partial x'^k} B_n = \det M \frac{\partial x'^i}{\partial x^r} \cdot \underbrace{\frac{\partial x^m}{\partial x'^j} \frac{\partial x'^j}{\partial x^s}}_{\delta^m{}_s}$$

$$\underbrace{\frac{\partial x^n}{\partial x'^k} \frac{\partial x'^k}{\partial x^t}}_{\delta^n{}_t} \varepsilon^{rst} A_m B_n$$

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$$= \det M \frac{\partial x'^i}{\partial x^r} \delta^m{}_s \delta^n{}_t \varepsilon^{rst} A_m B_n =$$

$$= \det M \frac{\partial x'^i}{\partial x^r} \underbrace{\varepsilon^{rmn} A_m B_n}_{C^r} = \boxed{\det M \frac{\partial x'^i}{\partial x^r} C^r}$$

pseudo vector.

Dual Tensors

In some cases it is possible to develop a one to one correspondence between two tensors of different ranks. In this case the tensor of lower rank is called the dual tensor.

Example:

C^{jk} is an antisymmetric tensor of rank 2 in 3D.

It has only 3 independent components that correspond to the number of components of a tensor of rank 1 in 3D.

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Let's define:

$$C_i = \frac{1}{2} \varepsilon_{ijk} C^{jk}$$

$$C^{jk} = \begin{pmatrix} 0 & C^{12} & C^{13} \\ -C^{12} & 0 & C^{23} \\ -C^{13} & -C^{23} & 0 \end{pmatrix}$$

$$C_1 = \frac{1}{2} \varepsilon_{1jk} C^{jk} = \frac{1}{2} \left(\varepsilon_{123} C^{23} + \varepsilon_{132} C^{32} \right) =$$

$$= \frac{1}{2} \left(1 \, C^{23} + (-1)(-C^{23}) \right) = C^{23}$$

You can see that $C_2 = C^{13}$ and $C_3 = C^{12}$

Notice that if C^{jk} is a tensor then C_i is a pseudovector.

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Then $C^i = (C^{23}, C^{13}, C^{12})$

Note: All the vectors in 3D physics that arise from a cross product of vectors are really duals of antisymmetrized direct products of the vectors that are cross-multiplied.

$$\bar{L} = \bar{r} \times \bar{p}$$

$$L_i = \varepsilon_{ijk} r^j p^k = \frac{1}{2} \varepsilon_{ijk} (r^j p^k - r^k p^j)$$

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Notice that if $N \neq 3$ this is not true:

$N = 2$

$$T^{ij} = S^i R^j$$

$$A^{ij} = S^i R^j - S^j R^i = T^{ij} - T^{ji}$$

$$A^{ij} = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}$$

one independent component

but a vector in $N=2$ has

two components $V^i = (V^1, V^2)$.

Exercise: do the same for $N=4$.

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Other uses of dual tensors:

1) consider $V^{ijk} = A^i B^j C^k$ A, B, C, vectors

Define:

$$V = \varepsilon_{ijk} V^{ijk} = \varepsilon_{ijk} A^i B^j C^k =$$

$$= A^i \underbrace{\varepsilon_{ijk} B^j C^k}_{(\bar{B}\times\bar{C})_i} = \bar{A} \cdot (\bar{B} \times \bar{C})$$

V is a pseudoscalar because A, B, C are vectors but ε_{ijk} is a pseudotensor.

volume of the cell expanded by $\bar{A}$, $\bar{B}$ and $\bar{C}$.

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Use of tensor notation to solve problems:

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = ?$$

$$\underbrace{\bar{\nabla} \times (\underbrace{\bar{\nabla} \times \bar{A}}_{\varepsilon_{ijk}\,\partial^j A^k})}_{\varepsilon^{rmi}\,\partial_m\,\varepsilon_{ijk}\,\partial^j A^k}$$

r — free index

I'll obtain V^r

$$\varepsilon^{rmi}\partial_m \varepsilon_{ijk}\partial^j A^k = \underbrace{\varepsilon^{rmi}\varepsilon_{ijk}}_{\underbrace{\varepsilon^{irm}\varepsilon_{ijk}}_{\delta^r{}_j\delta^m{}_k - \delta^r{}_k\delta^m{}_j \text{ (homework)}}}\partial_m\partial^j A^k =$$

$$= (\delta^r{}_j\delta^m{}_k - \delta^r{}_k\delta^m{}_j)\partial_m\partial^j A^k =$$

$$= \delta^r{}_j\delta^m{}_k\partial_m\partial^j A^k - \delta^r{}_k\delta^m{}_j\partial_m\partial^j A^k =$$

$$= \underbrace{\partial_k\partial^r A^k}_{\partial^r\partial_k A^k} - \partial_j\partial^j A^r = V^r$$

Then

$$\bar{V} = \bar{\nabla}.(\bar{\nabla}.\bar{A}) - \bar{\nabla}^2\bar{A}$$

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