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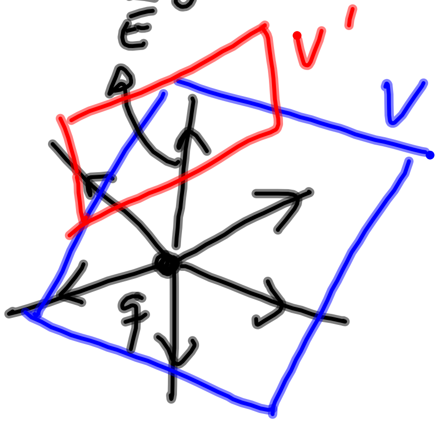
Divergence:

A vector field $\bar{G}(\bar{x})$ has non-zero divergence in a volume V if it has a singularity inside V:



$$\bar{\nabla}\cdot\bar{E} \propto q$$

In V $\bar{\nabla}\cdot\bar{E} \neq 0$

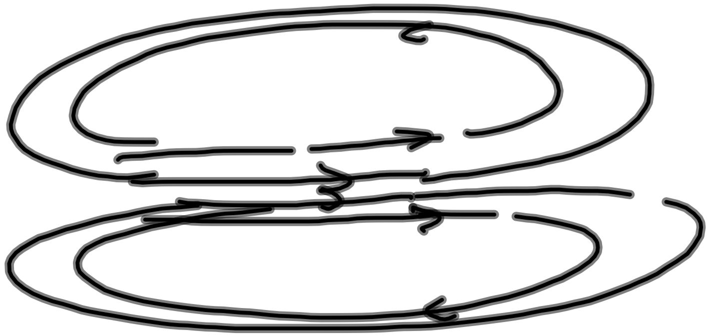
but in V'

$$\bar{\nabla}\cdot\bar{E} = 0$$ (no charge, or singularity, in V').

$\bar{G} = \bar{E}$ in this example.

If $\bar{\nabla}\cdot\bar{G} = 0$ everywhere $\bar{G}$ is solenoidal. Example: $\bar{B}$ (magnetic field).

$\bar{\nabla} \cdot \bar{B} = 0$ since there are not magnetic monopoles.



Solenoidal field
(no divergencies).

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Useful expressions valid mostly in 3D.

1. $\bar{\nabla}\cdot\bar{r} = \bar{\nabla}\cdot(x\hat{x} + y\hat{y} + z\hat{z}) = 3$

$\overset{\shortparallel}{\partial_i x^i}$ $\quad (\partial_i x^i = N$ in dimension $N)$

2. $\bar{\nabla}\cdot(\bar{r} f(r)) = \sum_i \frac{\partial}{\partial x^i}[x^i f(r)] =$

magnitude of $\bar{r}$

$r = \left[\sum_i x^{i\,2}\right]^{1/2}$

$\frac{\partial r}{\partial x^i} = \frac{1}{2}\frac{2x^i}{\left[\sum_i x^{i\,2}\right]^{1/2}} = \frac{x^i}{r}$

$= \sum_i \left[\underbrace{\frac{\partial x^i}{\partial x^i}}_{1} f(r) + x^i \frac{\partial f(r)}{\partial x^i}\right] = \sum_i f(r) + \sum_i x^i \frac{\partial f(r)}{\partial x^i}$

$= 3f(r) + \sum_i x^i \frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x^i} = 3f(r)\ \sum_i x^i \frac{\partial f(r)}{\partial r}\frac{x^i}{r} =$

$$= 3f(r) + \frac{1}{r}\frac{\partial f(r)}{\partial r}\underbrace{\sum_i x_i^2}_{r^2} = \boxed{3f(r) + r\frac{\partial f(r)}{\partial r}}$$

using 2)

$$3)\quad \bar{\nabla}\cdot(\bar{r}\underbrace{r^{n-1}}_{f(r)}) = 3r^{n-1} + r\frac{\partial r^{n-1}}{\partial r} = 3r^{n-1} + (n-1)r\,r^{n-2}$$

$$= 3r^{n-1} + (n-1)r^{n-1} = \boxed{(n+2)r^{n-1}}$$

Curl

$$\bar{\nabla} \times \bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} +$$

vector field

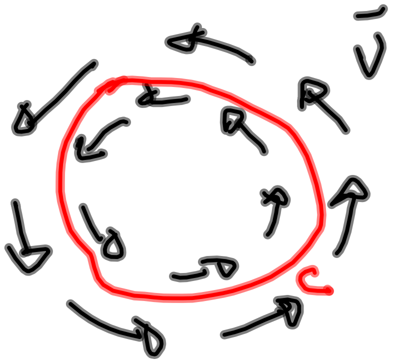
$$+ \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

Examples:

$\bar{B} = \bar{\nabla} \times \bar{A}$ ($\bar{A}$ vector potential)

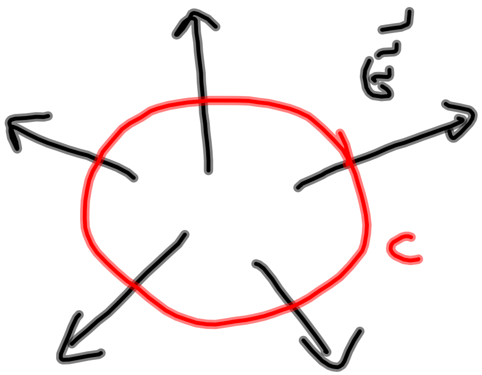
$\bar{\nabla} \times \bar{E} = 0$ for electrostatic field.

Physical interpretation of the curl:



If a field $\bar{V}$ has a finite circulation along a close line c then $\bar{\nabla}\times\bar{V}\neq 0$

$$\oint_c \bar{V}\cdot d\bar{l} \neq 0$$



$$\oint_c \bar{E}\cdot d\bar{l} = 0$$

Example:

$$\bar{\nabla} \times (\bar{r} f(r)) = ?$$

I'm going to use an expression from the back of the book that we are going to demonstrate later using tensors.

$$\bar{\nabla} \times (f \bar{V}) = f \bar{\nabla} \times \bar{V} + (\bar{\nabla} f) \times \bar{V}$$

In our case

$$\bar{\nabla} \times (f(r) \bar{r}) = f(r) \underbrace{\bar{\nabla} \times \bar{r}}_{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = 0} + \underbrace{(\bar{\nabla} f) \times \bar{r}}_{0} = 0$$

since $f = f(r) \Rightarrow \bar{\nabla} f \parallel \bar{r}$

Then $\bar{r} f(r)$ is irrotational.

Laplacian: $\bar{\nabla}^2 = \bar{\nabla}\cdot\bar{\nabla}$ scalar

$$\underbrace{\nabla^2 \varphi}_{\text{scalar}} = \underbrace{\bar{\nabla}\cdot(\underbrace{\bar{\nabla}\varphi}_{\text{gradient: vector}})}_{\text{divergence: scalar}} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

Scalar

In tensor notation: $\bar{\nabla} = \partial_i$

$$\nabla^2 = \partial_i \partial^i \equiv \frac{\partial^2}{\partial x_i^2}$$

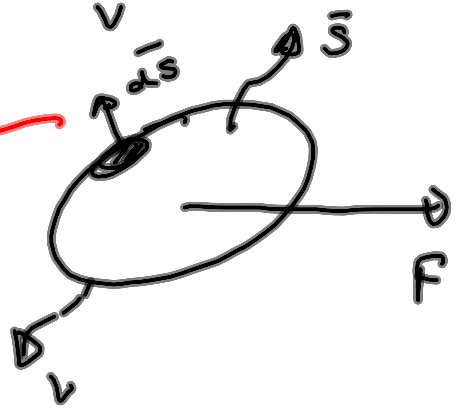
Examples: $\bar{E} = -\bar{\nabla}\phi$ and $\bar{\nabla}\cdot\bar{E} = 0$ (if there are no charges)

$0 = \bar{\nabla}\cdot\bar{E} = \bar{\nabla}\cdot(-\bar{\nabla}\phi) = -\nabla^2\phi \Rightarrow \nabla^2\phi = 0$ (Laplace's equation).

Divergence theorem or Gauss' Theorem:

$$\int_V \bar{\nabla} \cdot \bar{F} \, dV = \oint_S \bar{F} \cdot \bar{dS}$$

for V inside S.



$\bar{dS}$: magnitude of dS (surface differential) it points in the direction $\perp$ to S and outside the volume.

Examples : Gauss' law: $\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\varepsilon_0}$ (Maxwell's eq.)

$$\oint_S \bar{E} \cdot \bar{dS} = \int_V \bar{\nabla} \cdot \bar{E} \, dV = \int_V \frac{\rho}{\varepsilon_0} \, dV = \frac{q}{\varepsilon_0}$$

charge inside V

useful to get $\bar{E}$ when it's constant on a surface.

2) Poisson's equation:

$$\bar{\nabla} \cdot \bar{E} = \rho / \varepsilon_0 \qquad \text{Maxwell}$$

in electrostatics $\bar{E} = -\bar{\nabla}\phi$

$$\therefore \quad \bar{\nabla} \cdot \bar{E} = -\nabla^2 \phi = \rho/\varepsilon_0 \Rightarrow \boxed{\nabla^2 \phi = -\rho/\varepsilon_0}$$

Poisson's eq.

3) Ampère's law: (Stoke's theorem)

$$\int_S \bar{\nabla} \times \bar{H} \cdot d\bar{S} = \int \bar{J} \cdot d\bar{S} = \bar{I}_{\text{throug surface}}$$

$\bar{\nabla} \times \bar{H} = \bar{J}$ (Maxwell's eq.)

$\bar{I}$: current

$\bar{J} = \dfrac{\bar{I}}{A}$ $A \to$ area

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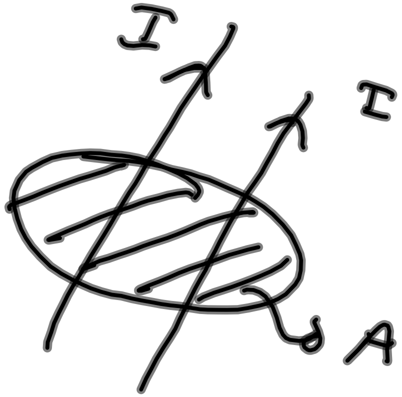
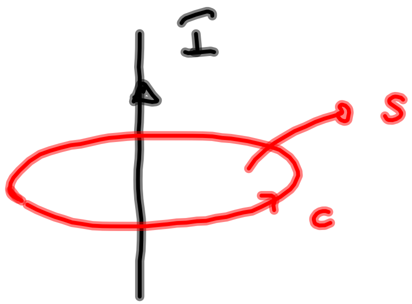


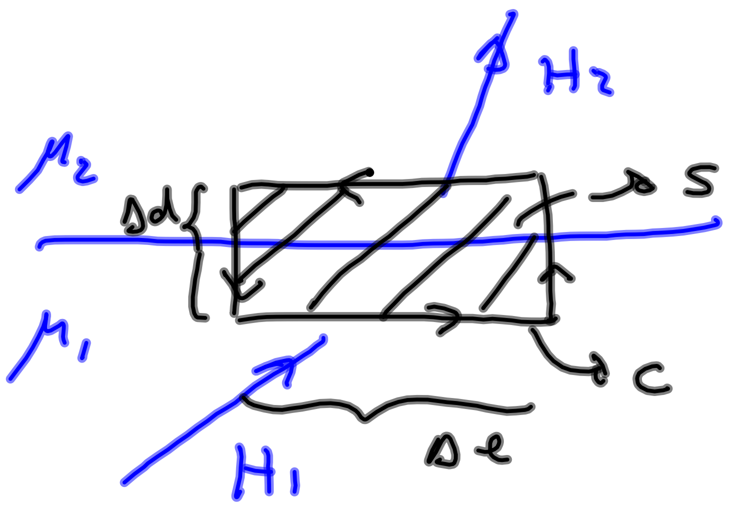
$\bar{J} = \dfrac{\bar{I}}{A}$ density of current per unit area.



$\oint_c \bar{H} \cdot d\bar{l} = \bar{I}$ through the surface.

Stokes theorem $\oint_c \bar{H} \cdot d\bar{l} = \oint_s \bar{\nabla} \times \bar{H} \cdot d\bar{s}$

For homework.



What happens with the tangential component of $\bar{H}$ across the surface?

$$\bar{K} = \lim_{\substack{\Delta d \to 0 \\ J \to \infty}} \bar{J} \cdot \Delta d$$

linear density of current

analogous to point like dipole:

$\bar{p} = \bar{d}\, q$ (real dipole)

$q \quad -q$

$\lim_{\substack{d \to 0 \\ q \to \infty}} \bar{d} q = p$ (point like dipole).

Dirac Delta Function

If $V = V(r)$ $\qquad r = \sqrt{x^2+y^2+z^2}$

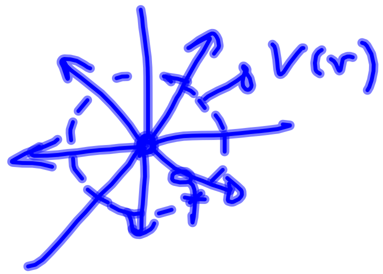
$\bar{\nabla} V = \frac{\partial V}{\partial r}\hat{r}$ gradient $\perp$ to equipotentials.

Consider a point charge $q = 4\pi\varepsilon_0$ ⊛ at the origin.

$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\varepsilon_0}$ $\qquad \bar{E} = -\bar{\nabla}\phi$

$$\boxed{\bar{E} = \frac{q}{4\pi\varepsilon_0 r^2}}\hat{r} \overset{⊛}{=} \frac{4\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} = \boxed{\frac{\hat{r}}{r^2}}$$

$$\boxed{\phi = \frac{q}{4\pi\varepsilon_0 r}} \doteq \boxed{\frac{1}{r}}$$



Let's use Gauss' theorem:

$$\oint_S \bar{E}\cdot d\bar{S} = \int_V \bar{\nabla}\cdot\bar{E}\, dV = \int_V \frac{\rho(\bar{x})}{\varepsilon_0}\, dV = \frac{q_{encl}}{\varepsilon_0} \begin{cases} \frac{4\pi\varepsilon_0}{\varepsilon_0} \\ \text{or} \\ 0 \end{cases}$$

4π if q is inside V or 0 if not.

also

$$\oint_S \bar{E}\cdot d\bar{S} = \oint \frac{\hat{r}}{r^2}\cdot d\bar{S} = \int_V \bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) dV =$$

$$= \int_V \bar{\nabla}\cdot\left(-\bar{\nabla}\left(\frac{1}{r}\right)\right) dV = -\int \nabla^2\left(\frac{1}{r}\right) dV$$

We can express the above result by introducing the delta "function" so that

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\bar{r}) = -4\pi\,\delta(x)\,\delta(y)\,\delta(z)$$

with

$$\delta(x) = 0 \quad \text{if} \quad x \neq 0$$

$$f(0) = \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx\,.$$

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