Dirac's Delta

9/13

We defined that

$$\nabla^2\left(\frac{1}{r}\right) \equiv -4\pi\,\delta(\bar{r}) = -4\pi\,\delta(x)\,\delta(y)\,\delta(z)$$

so that

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV = \begin{cases} -4\pi & \text{if the origin is in } V. \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$\nabla^2\left(\frac{1}{r}\right) = \begin{cases} 0 & \text{for } r \neq 0 \\ \text{Diverges} & \text{for } r = 0. \end{cases}$$

Sep 13-2:11 PM

Properties:

- $\delta(x) = 0$ if $x \neq 0$
- $\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0)$

If $f(x) = 1$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = \int_{-\infty}^{\infty} 1\, \delta(x)\, dx = f(0) = 1$$

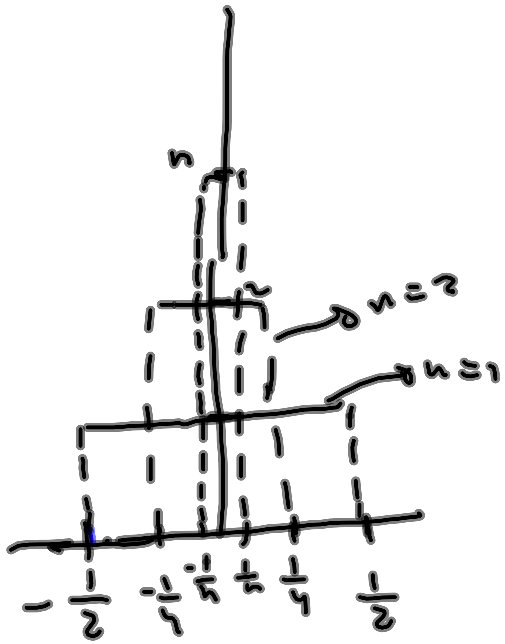
Then $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$, it is normalized.

Sep 13-2:15 PM

Notice that $\delta(x)$ is NOT a function. It is called a distribution. It can be obtained as the limit of a sequence of functions called support functions.

Examples of support functions:

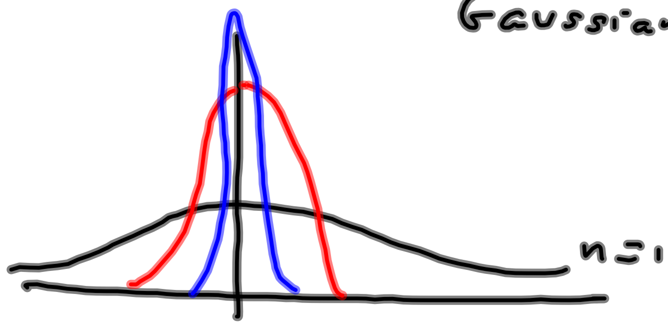
$$\delta_n(x) = \begin{cases} 0 & \text{if } x < -\frac{1}{2n} \\ n & \text{if } -\frac{1}{2n} \leq x \leq \frac{1}{2n} \\ 0 & \text{if } x > \frac{1}{2n} \end{cases}$$



Sep 13-2:18 PM

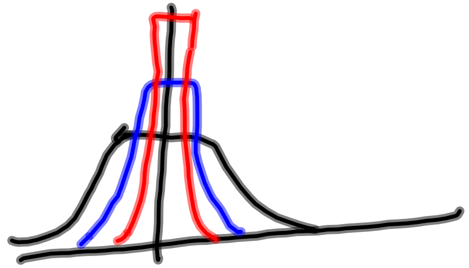
$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$$

Gaussians

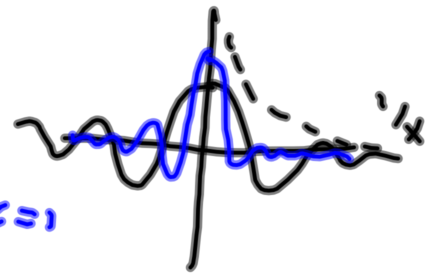


Lorentzians:

$$\delta_n(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2}$$



$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} e^{-ixt}\,dt$$



All support functions are normalized to 1: $\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1$

Notice that

$\lim_{n\to\infty} \delta_n(x)$ does not exist

(Physicist though would say $\lim_{n\to\infty} \delta_n(x) = \delta(x)$)

What exists is

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx = f(0) \qquad 1$$

Ex:

$$\lim_{n\to\infty} \int_{-1/2n}^{1/2n} n f(x)\, dx = \lim_{n\to\infty} n \underbrace{\int_{-1/2n}^{1/2n} f(x)\, dx}_{f(0)\Delta = f(0)\frac{1}{n}} = \lim_{n\to\infty} n f(0) \frac{1}{n} = f(0)$$

Now from ① we can write:

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx = f(0)$$

More properties:

- $\displaystyle\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx \underset{\substack{y = x-a \\ dy = dx}}{=} \int_{-\infty}^{\infty} f(y+a)\,\delta(y)\,dy = f(a)$

- $$\int_{-\infty}^{\infty} f(x)\,\delta(g(x))\,dx = \int_{-\infty}^{\infty} f(x) \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\,dx$$

x_i are the zeroes of $g(x)$: $g(x_i)=0$

and $g'(x_i) = \left.\frac{dg}{dx}\right|_{x=x_i}$ if $g'(x_i) \neq 0$.

Ex:

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx = \int_{-\infty}^{\infty} f(x)\,\frac{\delta(x)}{|a|}\,dx = \frac{f(0)}{|a|}$$

$g(x) = ax$ $x_1 = 0$ is such that $g(x_1)=0$

$g'(x) = a$ $g'(x_1=0) = a$

Sep 13-2:40 PM

Generalization beyond one dimension:

In 3D we define: (cartesian coordinates)

$$\delta(\bar{x}-\bar{X}) = \delta(x_1 - X_1)\,\delta(x_2 - X_2)\,\delta(x_3 - X_3)$$

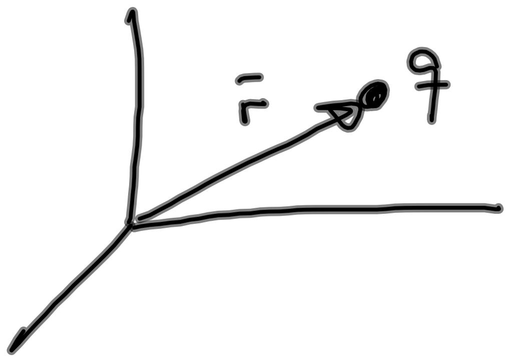
with

$$\int_V \delta(\bar{x}-\bar{X})\, d^3x = \begin{cases} 1 & \text{if } \bar{x} = \bar{X} \text{ and } \bar{X} \text{ is in } V. \\ 0 & \text{otherwise.} \end{cases}$$

- Mathematically $\delta(\bar{x})$ only has meaning in an integral. However, physicists use δ's to describe singularities.

Sep 13-2:45 PM

Ex: what is the charge density of a point-like charge q at $\bar{r}$?



$\rho(\bar{x}) = ?$ Density of charge in all space?

I know that

$$\int_{\text{all space}} \rho(\bar{x})\, dV = q$$

then I define $\rho(\bar{x}) = q\,\delta(\bar{x}-\bar{r})$ then

$$\int \rho(\bar{x})\, dV = \int q\,\delta(\bar{x}-\bar{r})\, dV = q$$

Sep 13-2:49 PM

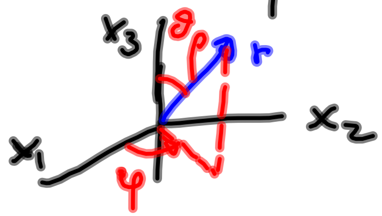
What system of coordinates should we use to express $\delta(\bar{r}-\bar{x})$?

This depends on the geometry of our problem.

If we are studying q inside a box we'll write:

$$\delta(\bar{r}-\bar{x}) = \delta(r_1-x_1)\,\delta(r_2-x_2)\,\delta(r_3-x_3)$$

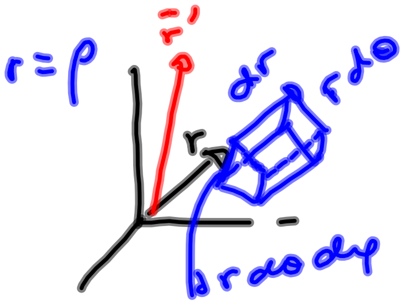
If we are studying q inside a sphere we want to write $\delta(\bar{r}-\bar{x})$ in terms of (ρ, θ, φ) (spherical coordinates)



Sep 13-2:54 PM

Let's express $\delta(\bar{r}-\bar{r}')$ in spherical coordinates

$\bar{r} = (\rho, \theta, \varphi)$



We know that $\int_V \delta(\bar{r})\,dV = 1$

$$1 = \int_0^\infty \rho^2 d\rho \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi\ \underbrace{\delta(\bar{r}-\bar{r}')}_{A\,\delta(\rho-\rho')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')} =$$

$$= \underbrace{\int_0^\infty \rho^2 d\rho\, \delta(\rho-\rho')}_{\rho'^2}\, A \underbrace{\int_0^\pi \delta(\theta-\theta') \sin\theta\, d\theta}_{\sin\theta'} \underbrace{\int_0^{2\pi} \delta(\varphi-\varphi')\, d\varphi}_{1}$$

δ is normalized in $(0, 2\pi)$

Then I need that $A = \frac{1}{\rho'^2 \sin\theta'} \overset{\text{or}}{\equiv} \frac{1}{\rho^2 \sin\theta}$

Then

$$\boxed{\delta(\bar{r} - \bar{r}') = \frac{1}{\rho^2 \sin\theta}\, \delta(\rho - \rho')\, \delta(\theta - \theta')\, \delta(\varphi - \varphi')}$$

In general if $x_i = f(\xi_i)$

$$\delta(\bar{x} - \bar{x}') = \frac{\delta(\xi_1 - \xi_1')\, \delta(\xi_2 - \xi_2')\, \delta(\xi_3 - \xi_3')}{|J(x_i, \xi_i)|}$$

Jacobian

Sep 13-3:14 PM

$$J(x_i;\xi_i) = \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix}$$

In spherical coordinates: $x_i = (x, y, z)$ $\xi_i = (\rho, \theta, \varphi)$

$$x = \rho \sin\theta \cos\varphi$$
$$y = \rho \sin\theta \sin\varphi$$
$$z = \rho \cos\theta$$

$$J(x,y,z;\rho,\theta,\varphi) = \begin{vmatrix} \sin\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\theta & -\rho\sin\theta & 0 \end{vmatrix} = \rho^2\sin\theta$$

Sep 13-3:17 PM

Notice that some times we use: $\bar{r} = (\rho, \cos\theta, \varphi)$

then $$\delta(\bar{r} - \bar{r}') = \frac{\delta(\rho - \rho')\,\delta(\cos\theta - \cos\theta')\,\delta(\varphi - \varphi')}{r^2}$$

Sep 13-3:23 PM