Other properties of $\delta(x)$ 9/18

- we can define an operator

$$\mathcal{L}(x_0) = \int dx\, \delta(x-x_0)$$

then

$$\mathcal{L}(x_0) f(x) = \int dx\, \delta(x-x_0) f(x) = f(x_0)$$

$\mathcal{L}$ allows to map the function from x to x_0.

- $\delta(x-x')$ or $\delta(x'-x)$ describes a singularity at $x=x'$.

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The δ can be expanded in terms of orthogonal functions.

If $\{\varphi_n(x)\}$ are a basis of orthogonal functions in the interval $[a,b]$ so that

$$\int_a^b \varphi_m^*(x)\,\varphi_n(x)\,dx = \delta_{mn}$$

→ orthonormal in this case

(any set of orthogonal functions can be normalized)

then any function and also the δ can be expressed in terms of $\{\varphi_n(x)\}$.

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Any well behaved function $f(x)$ can be expanded as:

$$f(x) = \sum_{n=0}^{\infty} b_n \varphi_n(x) \quad (1) \qquad \text{with} \quad b_n = \int_a^b f(x)\varphi_n^*(x)\,dx \quad (2)$$

D/ How to find b_m?

Multiply (1) by $\varphi_m^*(x)$ on both sides and integrate over x from a to b:

$$\int_a^b f(x)\varphi_m^*(x)\,dx = \sum_{n=0}^{\infty} b_n \underbrace{\int_a^b \varphi_m^*(x)\varphi_n(x)\,dx}_{\delta_{m,n}} = b_m$$

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Now instead of $f(x)$ let's use $\delta(x-t)$:

$$\delta(x-t) = \sum_{n=0}^{\infty} a_n \varphi_n(x) \quad ①$$

Let's find a_n:

$$\underbrace{\int_a^b \delta(x-t)\, \varphi_m^*(x)\, dx}_{} = \sum_{n=0}^{\infty} a_n \underbrace{\int_a^b \varphi_m^*(x)\, \varphi_n(x)\, dx}_{\delta_{m,n}}$$

$$\boxed{\varphi_m^*(t) = a_m} \quad ②$$

Plugging ② in ① we get:

$$\boxed{\delta(x-t) = \sum_{n=0}^{\infty} \varphi_n^*(t)\, \varphi_n(x)}$$

Ex:

$$\{\varphi_n(x)\} = \left\{\sqrt{\frac{2}{a}} \cos\frac{2n\pi x}{a}\right\}$$ orthonormal in $\left[-\frac{a}{2}, \frac{a}{2}\right]$

$$\frac{2}{a}\int_{-a/2}^{a/2} \cos\frac{2n\pi x}{a} \cos\frac{2m\pi x}{a}\, dx = \delta_{m,n}$$

$$\delta(x-t) = \sum_{n=0}^{\infty} a_n(t) \cos\frac{2\pi n x}{a}\sqrt{\frac{2}{a}}$$

$$\sqrt{\frac{2}{a}}\int_{-a/2}^{a/2} \cos\frac{2m\pi x}{a}\, \delta(x-t)\, dx = \frac{2}{a}\sum_{n=0}^{\infty} a_n(t) \underbrace{\int_{-a/2}^{a/2} \cos\frac{2\pi m x}{a} \cos\frac{2\pi n x}{a}\, dx}_{\frac{a}{2}\delta_{m,n}}$$

$$\sqrt{\frac{2}{a}} \cos\frac{2m\pi t}{a} = a_m(t)$$

$$\delta(x-t) = \frac{2}{a}\sum_{n=0}^{\infty} \cos\frac{2n\pi t}{a} \cos\frac{2n\pi x}{a}$$

Tensor analysis

A tensor of rank k in a space of dimension N has N^k components.

$k=0 \longrightarrow$ scalar $N^0 = 1$ (one component)

$k=1 \longrightarrow$ vector $N^1 = N$ (N components)

$k=2 \longrightarrow$ matrix N^2

$\vdots$

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When we go from a system K to a system K' the components of a tensor may change in one of two ways: contravariantly or covariantly.

Example: We learned before that A^i transforms as $A^{i'} = \frac{\partial x^{i'}}{\partial x^j} A^j$ contravariant

and

B_j transforms as $B'_j = \frac{\partial x^i}{\partial x'^j} B_i$ covariant.

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Direct Product or Outer Product.

We can use vectors to generate tensors of higher rank.

$$A^i B^j = C^{ij}$$

In $N=2$

$$C^{ij} = \begin{pmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{pmatrix} = \begin{pmatrix} A^1 B^1 & A^1 B^2 \\ A^2 B^1 & A^2 B^2 \end{pmatrix}$$

How does C^{ij} transform from K to K'?

$$C'^{ij} = A'^{i} B'^{j} = \frac{\partial x'^{i}}{\partial x^{k}} A^{k} \frac{\partial x'^{j}}{\partial x^{l}} B^{l} =$$

$$= \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x'^{j}}{\partial x^{l}} \underbrace{A^{k} B^{l}}_{C^{kl}} =$$

$$= \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x'^{j}}{\partial x^{l}} C^{kl}$$

Then we define C^{ij} as our prototype contravariant rank 2 tensor.

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Via the outer product of two tensors of rank 1 I obtained a tensor of rank 2.

Notice that

$$A_i B^j = C_i{}^j \qquad \text{mixed tensor of rank 2}$$

$$C'_i{}^j = \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} \underbrace{A_k B^l}_{C_k{}^l} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} C_k{}^l$$

$$A^i B_j = C^i{}_j \qquad \text{mixed tensor of rank 2 with}$$

$$C'^i{}_j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} C^k{}_l$$

The covariant tensor of rank 2 is given by

$$A_i B_j = C_{ij}$$

$$C'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} C_{kl}$$

Higher rank tensors:

They are obtained doing the direct product of lower rank tensors:

$$C^{ij} B^k = T^{ijk}$$

rank 3 - The rank of the result is the sum of the ranks of the tensors involved.

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$$T'^{ijk} = C'^{ij} B'^{k} = \frac{\partial x'^i}{\partial x^l} \frac{\partial x'^j}{\partial x^m} C^{lm} \frac{\partial x'^k}{\partial x^r} B^r =$$

$$= \frac{\partial x'^i}{\partial x^l} \frac{\partial x'^j}{\partial x^m} \frac{\partial x'^k}{\partial x^r} \underbrace{C^{lm} B^r}_{T^{lmr}}$$

Other Examples:

$$B^{i}{}_{j}{}^{k}\, T_{lm}{}^{n} = M^{i}{}_{j}{}^{k}{}_{lm}{}^{n}$$ rank 6 mixed tensor.

$M' \longrightarrow M$ via six derivative factors.

(Do it at home!)

Addition of tensors:

You can add tensors of the same rank and with indices that transform in the same way.

$$C^{ab} = A^{ab} + B^{ab}$$

$$C^{a}{}_{b} = A^{a}{}_{b} + B^{a}{}_{b}$$

$$A^{a}{}_{b} + B^{ab}$$ NOT a TENSOR!

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Contraction of tensors

If two indices of a tensor of rank n $(n \geq 2)$, one index covariant and the other contravariant, are repeated implying a sum, the 2 indeces are contracted and the result is a tensor of rank $k = n-2$.

Consider $C^i{}_j = \begin{pmatrix} C^1{}_1 & C^1{}_2 \\ C^2{}_1 & C^2{}_2 \end{pmatrix}$ $N = 2$ rank 2

Calculate $C^i{}_i = C^1{}_1 + C^2{}_2 =$ trace of $C^i{}_j$ scalar rank 0

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We see that $C^i{}_i$ is a scalar by transforming $C^i{}_i$ from S to S' (or vice versa):

$$C'^i{}_i = \frac{\partial x'^i}{\partial x^j} \frac{\partial x^k}{\partial x'^i} C^j{}_k =$$

$$= \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^i}{\partial x^j} C^j{}_k = \underbrace{\frac{\partial x^k}{\partial x^j}}_{\delta^k{}_j} C^j{}_k =$$

$$= \delta^k{}_j C^j{}_k = C^k{}_k$$

$C'^i{}_i = C^k{}_k$ it transforms as a scalar.

In general consider:

$$A^i{}_j B^{kl} = T^i{}_j{}^{kl}$$ tensor of rank 4

Then

$$A^i{}_i B^{kl} = T^i{}_i{}^{kl} = M^{kl}$$ tensor of rank 2.

I also can contract:

$$T^i{}_j{}^{jl} = N^{il}$$

$$T^i{}_j{}^{kj} = W^{ik}$$

I cannot contract
k and l
k and i
or
i and l

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Matrix Multiplication vs tensor contraction.

A tensor of rank 2 is a matrix.

Matrix multiplication is an special case of contraction:

$$\mathbb{C} = \mathbb{A} \cdot \mathbb{B}$$ matrix multiplication

you write:

$$C_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj}$$

What is this operation for matrices that are tensors?

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$$C^{il} = A^{i}{}_{k} B_{k}{}^{l} \quad \text{or} \quad C^{il} = A^{ik} B_{k}{}^{l}$$

$$C^{i}{}_{l} = A^{ik} B_{kl} \quad \text{or} \quad A^{i}{}_{k} B^{k}{}_{l}$$

$$C_{i}{}^{l} = A_{i}{}^{k} B_{k}{}^{l} \quad \text{or} \quad A_{ik} B^{kl}$$

$$C_{il} = A_{ik} B^{kl} \quad \text{or} \quad A_{i}{}^{k} B_{kl}$$

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