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Rank 2 tensors and Matrix Multiplication.

You can multiply rank 2 tensors as matrices if they have the correct indices:

- You can do: $C^{il} = A^{i}{}_{k} B^{kl}$

This is ok because in a system S' we'll get

$$C'^{il} = A'^{i}{}_{k} B'^{kl} = \frac{\partial x'^{i}}{\partial x^{r}} \underbrace{\overbrace{\frac{\partial x^{s}}{\partial x'^{k}} \frac{\partial x'^{k}}{\partial x^{t}}}^{\frac{\partial x^s}{\partial x^t} = \delta_{s,t}}} \frac{\partial x'^{l}}{\partial x^{u}} A^{r}{}_{s} B^{tu} =$$

$$= \frac{\partial x'^{i}}{\partial x^{r}} \frac{\partial x'^{l}}{\partial x^{u}} \underbrace{A^{r}{}_{s} B^{su}}_{C^{ru}} = \frac{\partial x'^{i}}{\partial x^{r}} \frac{\partial x'^{l}}{\partial x^{u}} C^{ru}$$

contravariant rank 2 tensor.

However you can have A^{ik} and B^{lj} and as matrices you can calculate in S

$$\sum_k A^{ik} B^{kj} = C^{ij}$$

this is a matrix but is it a tensor?

What happens in S':

$$C'^{ij} = \sum_k A'^{ik} B'^{kj} = \sum_k \frac{\partial x'^i}{\partial x^r} \frac{\partial x'^k}{\partial x^s} A^{rs} \cdot \frac{\partial x'^k}{\partial x^t} \frac{\partial x'^j}{\partial x^m} B^{tm} = \sum_k \frac{\partial x'^i}{\partial x^r} \frac{\partial x'^k}{\partial x^s} \frac{\partial x'^k}{\partial x^t} \frac{\partial x'^j}{\partial x^m} \underbrace{A^{rs} B^{tm}}_{\text{NOT } C^{rm}}$$

This matrix product does NOT produce a tensor.

All rank 2 tensors are matrices but
NOT all matrices are rank 2 tensors!

Similarity transformation and rank 2 tensors.

If you have a matrix $\mathbb{A}$ in system S you transform it to system S' by doing:

$$\mathbb{A}' = U \mathbb{A} U^{-1}$$

where U is the change of base matrix

$$U_{ij} = \frac{\partial x'_i}{\partial x_j}$$ when S and S' are both orthogonal, cartesian systems.
So we do not care about covariant or contravariant components.

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However for general S and S' systems

$$A = A^{ij} \text{ or } A^{i}{}_{j} \text{ or } A_{i}{}^{j} \text{ or } A_{ij}$$

The transformation of which of these 4 forms corresponds to the similarity transformation? Now $U^{i}{}_{j} = \frac{\partial x'^{i}}{\partial x^{j}}$ $(U^{-1})^{k}{}_{\ell} = \frac{\partial x^{k}}{\partial x'^{\ell}}$

$$A'^{i}{}_{j} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{\ell}}{\partial x'^{j}} A^{k}{}_{\ell} = U^{i}{}_{k} (U^{-1})^{\ell}{}_{j} A^{k}{}_{\ell}$$

$$= U^{i}{}_{k} A^{k}{}_{\ell} (U^{-1})^{\ell}{}_{j}$$

similarity transformation

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Now let's transform A^{ij}

$$A'^{ij} = \frac{\partial x'^i}{\partial x^l} \frac{\partial x'^j}{\partial x^k} A^{lk} = U^i{}_l U^j{}_k A^{lk}$$

Notice that this means multiply rows of $U^i{}_l$ by columns of ~~A~~ · and columns of $U^j{}_k$ by columns of A^{lk}

But in matrix multiplication you always multiply rows by columns.

Ex: N=2

$$A'^{11} = U^1{}_1 U^1{}_1 A^{11} + U^1{}_2 U^1{}_1 A^{21} + U^1{}_1 U^1{}_2 A^{12} + U^1{}_2 U^1{}_2 A^{22}$$

Ex

$$A'_i{}^j = \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} A_k{}^l = (U^{-1})^k{}_i U^j{}_l A_k{}^l$$

This is not matrix multiplication

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Kronecker Delta

Is it a tensor? What kind?

Up to now we know:

$$\delta_{kl} = \begin{cases} 1 & \text{if } k=l \\ 0 & \text{if } k \neq l \end{cases}$$

in 2D as a matrix

$$\delta_{kl} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$$

identity matrix

δ_{kl} if it is a tensor will have rank 2.

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Now let's see how δ_{ij} transforms from S'
to S:

contravariant covariant

$$\delta'_{ij} = \frac{\partial x'_i}{\partial x'_j} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x_k}{\partial x'_j} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x_\ell}{\partial x'_j}\delta_{k\ell}$$

We notice that in tensor notation

$$\delta'^{i}{}_{j} = \frac{\partial x'^i}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^\ell}{\partial x'^j}\delta^{k}{}_{\ell}$$

mixed tensor of rank 2.

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Tensor Properties

In general for a tensor of rank 2

$$A^{mn} \neq A^{nm}$$

A^{mn} and A^{nm} are in general unrelated to each other.

However consider a tensor B with

if $B^{mn} = B^{nm}$ then B is symmetric.

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if for a tensor C we see that

$C^{mn} = -C^{nm}$ then C is antisymmetric.

Notice that $C^{mm} = -C^{mm}$ $\therefore$ $C^{mm} = 0$.

Independent components:

If a tensor is symmetric the number of independent components is smaller than the total number of components.

Ex: $N=2$ $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ only 3 indep. components.

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In general a symmetric tensor of rank 2 has N^2 components, but the number of independent components is

$$N + (N-1) + (N-2) + \cdots + 1 = \frac{N(N+1)}{2}$$

If $N = 3$

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{matrix}$$

For an antisymmetric matrix the number of independent components is

$$\frac{N(N+1)}{2} - N = \frac{N(N-1)}{2}$$

because $a_{ii}=0$

If $N=3$

$$\begin{matrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{matrix}$$

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Any rank 2 tensor can be written as the sum of a symmetric and an antisymmetric tensor:

$$A = A_s + A_a$$

$$A^{mn} = \underbrace{\tfrac{1}{2}\left(A^{mn} + A^{nm}\right)}_{A_s} + \underbrace{\tfrac{1}{2}\left(A^{mn} - A^{nm}\right)}_{A_a}$$

A_a is traceless.

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Higher rank tensors:

A tensor of rank k in N dimensions has N^k components. However, the number of independent components may be smaller if the tensor is symmetric or antisymmetric under the exchange of two indices.

If $T^{ijk\ell} = T^{jik\ell}$ T is symmetric under the exchange of i and j.

In 3D:
$3^4 = 81$ components

Since there are only 6 independent (ij) for each of the 9 $(k\ell)$ we get $9 \times 6 = 54$ independent components.

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In HW:

$$R_{iklm} = -R_{ikml} = -R_{kilm}$$

how many independent components are left?

Example: Stress Tensor

Hooks law: $\varepsilon = s\,\sigma$ or $\sigma = c\,\varepsilon$

(isotropic material)

ε: strain (deformation)

σ: stress (force)

c: $1/s$

For anisotropic materials:

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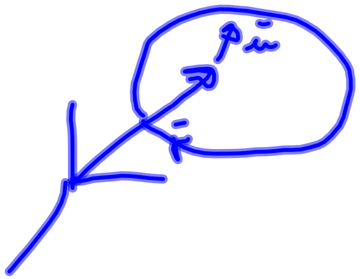
Strain tensor:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)$$

$\bar{u} = (u_x, u_y, u_z)$ deformation



Then $\sigma = \sigma_{\alpha\beta}$ tensor of rank 2

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\, \varepsilon_{\gamma\delta}$$

rank 4 elasticity tensor.

C has $3^4 = 81$ components.

Since $\varepsilon_{ij} = \varepsilon_{ji}$ and $\sigma_{ab} = \sigma_{ba}$

Then

$$C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\gamma\delta}$$

and

$$C_{\alpha\beta\gamma\delta} = C_{\alpha\beta\delta\gamma}$$

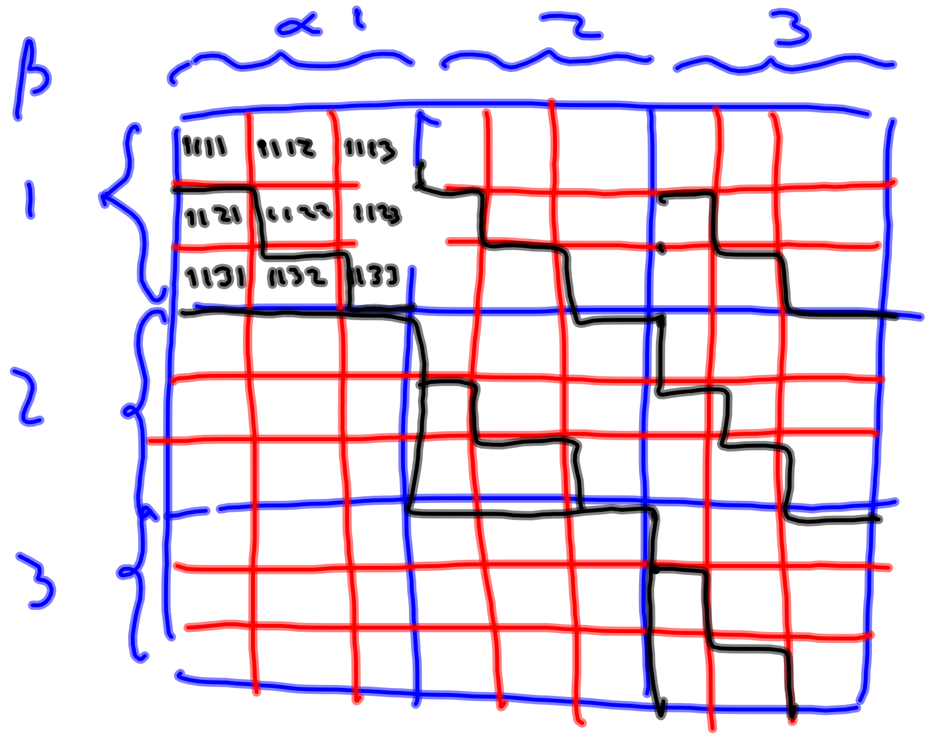
$\alpha\beta$ have 6 independent values

$\gamma\delta$ " " " "

So there are 36 independent components.

Also $C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta}$ this leaves 21 independent variables

Nice way of representing $C_{\alpha\beta\gamma\delta}$



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