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Midterm Oct. 2: Part I due in class.

Part II: due on October 9 (in class).

Homework #5 (does NOT enter in the test),
Posted today, due October 9.
(Work on it this week).

Tips for test: Review how to transform tensors from one system to another. Consider specific components: Ex: $r = (2,5)$ in orthogonal system.
Know how to find $r'^{i} = (x'^{1}, x'^{2})$ in S' and $r'_{i} = (x'_{1}, x'_{2})$ in S',

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Examples:

For homework you calculated

$A^{ij} = V^i W^j$

$V^i = (1, 2, 3)$

$W^j = (-1, 0, 4)$

$A^{ij} = ?$

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Quotient rule

If A_{ij} and B_{kl} are tensors we know that

$$A_{ij}\,B_{kl} = K_{ijkl}$$ is a tensor.

Now we want to know if K is a tensor knowing that A and B are tensors in situations like these:

$$K_i\,A^i = B \qquad K_i{}^j A_{jk} = B_{ik}$$

$$K_i{}^j A_j = B_i \qquad K^{ij} A^k = B^{ijk}$$

is K a tensor?

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In physical systems we may have a quantity in system S that is a tensor and we want to relate it to another tensor through an equation valid in system S. The right equation has to hold in any other system so we may need to find out whether some entity that relates two known tensors is a tensor.

Ex: Consider p_i : polarization vector in S.

E_j : electric field in S.

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In S we find that

$$p_i = P_{ij} E^j \quad (*) \quad \text{valid in } S.$$

is P_{ij} a tensor?

let's write the equation in S':

$$p'_i = \frac{\partial x^j}{\partial x'^i} p_j \overset{(*)}{=} \frac{\partial x^j}{\partial x'^i} (P_{jm} E^m) \underset{(*)}{=} \frac{\partial x^j}{\partial x'^i} P_{jm} \frac{\partial x^m}{\partial x'^r} E'^r$$

$$\| $$

$$P'_{ir} E'^r$$

$$E'^r = \frac{\partial x'^r}{\partial x^m} E^m$$

$$E^m = \frac{\partial x^m}{\partial x'^r} E'^r \quad (*)$$

Then $P'_{ir} E'^r = \dfrac{\partial x^j}{\partial x'^i} \dfrac{\partial x^m}{\partial x'^r} P_{jm} E'^r$

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Now I get that

$$\underbrace{\left(P'_{ir} - \frac{\partial x^j}{\partial x'^i}\frac{\partial x^m}{\partial x'^r} P_{jm}\right)}_{0} E'^r = 0$$

E'^r → arbitrary tensor

$$\therefore \quad P'_{ir} = \frac{\partial x^j}{\partial x'^i}\frac{\partial x^m}{\partial x'^r} P_{jm}$$

transforms as a covariant tensor of rank 2.

If we know that B is a tensor (ϕ_i in our example) and we want to know wheter K is a tensor (P_{ij} in ex.) we construct B by letting K act on an arbitrary tensor A (E^j in our ex). If the relationship holds in any system S'

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then K is a tensor.

If in S

$$\underbrace{K^{ij}}_{\text{tensor?}} \underbrace{A_j}_{\text{arbitrary tensor}} = \underbrace{B^i}_{\text{known tensor}} \qquad (*)$$

See whether $(*)$ holds in S' and find the transformation rules for K.

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Examples of tensors:

In 2D, rank 1:

$$V^k = \partial_j T^{jk} = \partial_1 T^{1k} + \partial_2 T^{2k} =$$

$$= \frac{\partial T^{1k}}{\partial x_1} + \frac{\partial T^{2k}}{\partial x_2}$$

$$\bar{V} = \left(\frac{\partial T^{11}}{\partial x_1} + \frac{\partial T^{21}}{\partial x_2}, \frac{\partial T^{12}}{\partial x_1} + \frac{\partial T^{22}}{\partial x_2} \right)$$

What can T^{jk} be?

Ex: $T^{jk} = \begin{pmatrix} x_1 + x_2 & x_1^2 - x_2 \\ x_1 & x_2 \end{pmatrix}$

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rank 2:

$$U^{ijk} T_{jk\ell} = C^{i}{}_{\ell} = \begin{pmatrix} C^1{}_1 & C^1{}_2 \\ C^2{}_1 & C^2{}_2 \end{pmatrix} =$$

rank 3 rank 3 rank 2

$$= \begin{pmatrix} U^{1jk} T_{jk1} & U^{1jk} T_{jk2} \\ U^{2jk} T_{jk1} & U^{2jk} T_{jk2} \end{pmatrix}$$

$$U^{1jk} T_{jk1} = U^{111} T_{111} + U^{121} T_{211} +$$

$$+ U^{112} T_{121} + U^{122} T_{221}$$

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Lowering and raising indices:

Fundamental tensor: metric tensor.

g_{ij} is a tensor that will allow us to lower indices of any tensor in a system S. Its inverse g^{ij} will allow us to raise indices of any tensor in S. g_{ij} can be an arbitrary tensor of rank 2 with $\det g_{ij} \neq 0$ (so that it has an inverse). But we are going to use as g_{ij} the so called "metric tensor".

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Let's consider a system S with coordinates $\{x^i\}$ and a system S' with coordinates $\{x'^i\}$.

In S let's have $d\bar{r} = dr^k$ a vector.

In S' we have $d\bar{r}' = dr'^k$ same vector.

We know that $d\bar{r} = d\bar{r}'$ (only the components get transformed)

$$d\bar{r} = x^i \bar{\varepsilon}_i \quad \text{and} \quad d\bar{r}' = x'^i \bar{\varepsilon}'_i$$

(1) (2)

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$\bar{\varepsilon}_j$ is a set of basis vectors in S and
$\bar{\varepsilon}'_j$ is a set of basis vectors in S'.

In cartesian coordinates $\bar{\varepsilon}_j = \hat{e}_j$ (versors)
but in general $|\bar{\varepsilon}_j| \neq 1$ (remember dual or contravariant basis vectors in oblique system!)

From (1) and (2) we obtain:

$$\bar{\varepsilon}_j = \frac{d\bar{r}}{dx^j} \quad \text{and} \quad \bar{\varepsilon}'_j = \frac{d\bar{r}'}{dx'^j}$$

Now let's define a rank 2 tensor g_{ij} through the direct (or outer) product of the basis vectors: In S:

labels component of the basis vector.

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j \equiv (\bar{\varepsilon}_i)_k (\bar{\varepsilon}_j)^k$$

labels basis vector

In cartesian coordinates

if S is cartesian:

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j = \frac{\partial \bar{r}}{\partial x^i} \cdot \frac{\partial \bar{r}}{\partial x^j} = \hat{e}_i \cdot \hat{e}_j$$

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$$g_{ij} = \hat{e}_i \cdot \hat{e}_j = \begin{pmatrix} \hat{e}_1 \cdot \hat{e}_1 & \hat{e}_1 \cdot \hat{e}_2 & \hat{e}_1 \cdot \hat{e}_3 \\ \hat{e}_2 \cdot \hat{e}_1 & \hat{e}_2 \cdot \hat{e}_2 & \hat{e}_2 \cdot \hat{e}_3 \\ \hat{e}_3 \cdot \hat{e}_1 & \hat{e}_3 \cdot \hat{e}_2 & \hat{e}_3 \cdot \hat{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

identity matrix.

Now let's consider how g_{ij} transforms:

In S:

$$ds^2 = d\bar{r} \cdot d\bar{r} = \frac{\partial \bar{r}}{\partial x^i} dx^i \cdot \frac{\partial \bar{r}}{\partial x^j} dx^j =$$

scalar

$$= \underbrace{\frac{\partial \bar{r}}{\partial x^i}}_{\bar{\varepsilon}_i} \cdot \underbrace{\frac{\partial \bar{r}}{\partial x^j}}_{\bar{\varepsilon}_j} dx^i dx^j = \underbrace{\bar{\varepsilon}_i \cdot \bar{\varepsilon}_j}_{g_{ij}} dx^i dx^j$$

tensor of rank 2

Since ds^2 is a tensor (rank 0) and dx^i are tensors of rank 1 if

$ds^2 = g_{ij}\, dx^i dx^j$ holds in all systems then g_{ij} has to be a tensor of rank 2.

Let's define another tensor g^{ij} so that

$g^{ik} g_{kj} = \delta^i{}_j$ then $g^{ik} = (g_{ik})^{-1}$

g^{ik} and g_{ik} are inverse of each other.

Now define that:

$$g^{ik} A_k = A^i \quad (*) \quad \text{(definition)}$$

then

$$g_{ik} A^k \overset{(*)}{=} g_{ik} \left[g^{kl} A_l \right] = \underbrace{g_{ik}\, g^{kl}}_{\delta_i^{\ l}} A_l =$$

$$= \delta_i^{\ l} A_l = A_i$$

We see that

$$g_{ik} A^k = A_i \quad \text{(lowers the index).}$$

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In S we know that

$$g^{ij} g_{jk} = \delta^i{}_k \qquad (1)$$

In S':

$$g'^{ij} g'_{jk} = \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^l} g^{ml} \frac{\partial x^r}{\partial x'^j} \frac{\partial x^s}{\partial x'^k} g_{rs} =$$

$$= \frac{\partial x'^i}{\partial x^m} \underbrace{\frac{\partial x^r}{\partial x'^j} \frac{\partial x'^j}{\partial x^l}}_{\frac{\partial x^r}{\partial x^l} = \delta^r{}_l} \frac{\partial x^s}{\partial x'^k} g^{ml} g_{rs} =$$

$$= \frac{\partial x'^i}{\partial x^m} \frac{\partial x^s}{\partial x'^k} g^{ml} g_{ls} = \frac{\partial x'^i}{\partial x^m} \frac{\partial x^s}{\partial x'^k} \delta^m{}_s = \frac{\partial x'^i}{\partial x'^k} = \delta'^i$$

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We see that 1 holds in S' then in S':

$$g'^{ij} g'_{jk} = \delta'^{i}{}_{k} .$$

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