Metric Tensor

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Last time we found that

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j$$

$$g_{ij}\, g^{jk} = \delta_i{}^k$$

$$(g_{ij})^{-1} = g^{ij}$$

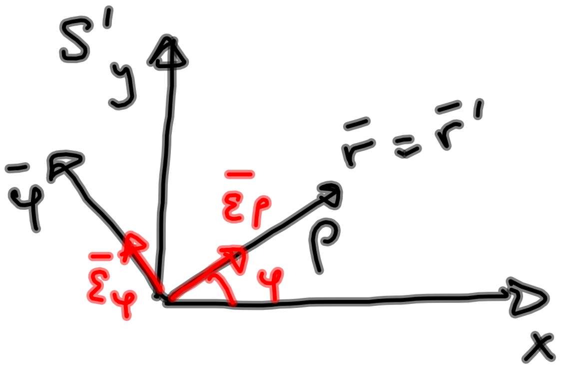
and also

$$g_{ij} A^j = A_i \quad \text{and} \quad g^{ij} A_j = A^i$$

Examples: Metric tensor in Polar Coordinates.

$$S : (x^1, x^2) = (\rho, \varphi)$$

$$S' = (x'^1, x'^2) = (x, y)$$



We need $x^j = x^j(x'^i)$

$\bar{r}' = x\,\hat{e}_1 + y\,\hat{e}_2$

$x'^i(x^j)$: $\qquad x^i(x'^j)$

$x = \rho\cos\varphi \qquad \rho = (x^2+y^2)^{1/2}$

$y = \rho\sin\varphi \qquad \varphi = \tan^{-1}\left(\frac{y}{x}\right)$

$$\bar{r}' = (x, y) = (\rho\cos\varphi, \rho\sin\varphi) = \rho\,\bar{\varepsilon}_\rho + \varphi\,\bar{\varepsilon}_\varphi \underset{(*)}{=} \bar{r}$$

$\bar{\varepsilon}_\rho = \frac{\partial \bar{r}}{\partial \rho}$ and $\bar{\varepsilon}_\varphi = \frac{\partial \bar{r}}{\partial \varphi}$ from (*)

Then

$$\bar{\varepsilon}_\rho = \frac{d\bar{r}}{d\rho} = \frac{d\bar{r}'}{d\rho} = (\cos\varphi, \sin\varphi) \qquad |\bar{\varepsilon}_\rho| = 1$$

$$\bar{\varepsilon}_\varphi = \frac{d\bar{r}}{d\varphi} = \frac{d\bar{r}'}{d\varphi} = (-\rho\sin\varphi, \rho\cos\varphi) \qquad |\bar{\varepsilon}_\varphi| = \rho \neq 1$$

Then:

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j = \begin{pmatrix} \bar{\varepsilon}_\rho \cdot \bar{\varepsilon}_\rho & \bar{\varepsilon}_\rho \cdot \bar{\varepsilon}_\varphi \\ \bar{\varepsilon}_\varphi \cdot \bar{\varepsilon}_\rho & \bar{\varepsilon}_\varphi \cdot \bar{\varepsilon}_\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}$$

diagonal because $\bar{\varepsilon}_\rho \perp \bar{\varepsilon}_\varphi$

In general, g_{ij} is symmetric because

$$\bar{\varepsilon}_i \cdot \bar{\varepsilon}_j = \bar{\varepsilon}_j \cdot \bar{\varepsilon}_i .$$

- g_{ij} is diagonal if the axis are orthogonal to each other.

$$g^{ij} = \left(g_{ij}\right)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\rho^2 \end{pmatrix}$$

Now we can obtain $\bar{\varepsilon}^i$ (contravariant or dual basis) for the polar system:

$$\bar{\varepsilon}^i = g^{ij}\,\bar{\varepsilon}_j \Rightarrow \begin{array}{l} \bar{\varepsilon}^\rho = g^{11}\,\bar{\varepsilon}_\rho = \bar{\varepsilon}_\rho \\ \bar{\varepsilon}^\varphi = g^{22}\,\bar{\varepsilon}_\varphi = \bar{\varepsilon}_\varphi/\rho^2 \end{array}$$

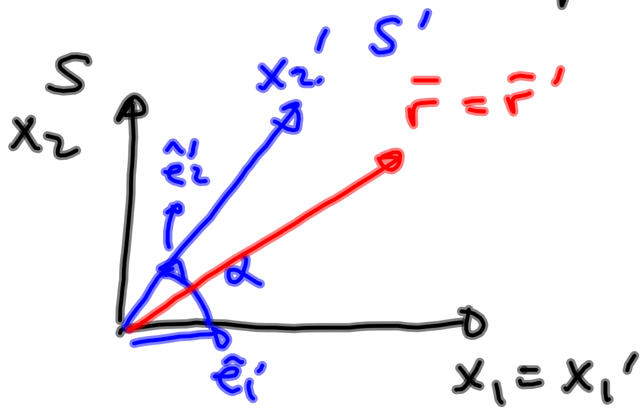
Notice that $|\bar{\mathcal{E}}_\varphi| = \rho$ $|\bar{\mathcal{E}}^\varphi| = 1/\rho$

but $\bar{\mathcal{E}}^\varphi \cdot \bar{\mathcal{E}}_\varphi = 1$ Invariant under a change of system of coordinates.

Example:

Oblique System.

$\hat{e}'_1 = \hat{e}_1 = (1,0)$
$\hat{e}'_2 = (\cos\alpha, \sin\alpha)$



we know that

$x_1 = x'^1 + x'^2 \cos\alpha$

$x_2 = x'^2 \sin\alpha$

In S $g_{ij} = \mathbb{I}$

In S' $g'_{ij} = \bar{\mathcal{E}}'_i \cdot \bar{\mathcal{E}}'_j = \hat{e}'_i \cdot \hat{e}'_j$

$$g'_{ij} = \hat{e}'_i \cdot \hat{e}'_j = \begin{pmatrix} \hat{e}'_1 \cdot \hat{e}'_1 & \hat{e}'_1 \cdot \hat{e}'_2 \\ \hat{e}'_2 \cdot \hat{e}'_1 & \hat{e}'_2 \cdot \hat{e}'_2 \end{pmatrix} = \qquad \begin{matrix} \hat{e}'_1 = (1, 0) \\ \hat{e}'_2 = (\cos\alpha, \sin\alpha) \end{matrix}$$

$$= \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}$$

$$g'^{ij} = (g'_{ij})^{-1} = \frac{1}{\sin^2\alpha} \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}$$

Notice that

$$x'_i = g_{ij} x^{i'}$$

$$x'_1 = g_{11} x'^1 + g_{12} x'^2 = x'^1 + x'^2 \cos\alpha$$

$$x'_2 = g_{21} x'^1 + g_{22} x'^2 = \cos\alpha \, x'^1 + x'^2$$

$$ds'^2 = d\bar{r}' \cdot d\bar{r}' = \bar{\varepsilon}'_j \, dx'^j \cdot \bar{\varepsilon}'_i \, dx'^i =$$

$$= \bar{\varepsilon}'_j \cdot \bar{\varepsilon}'_i \, dx'^j dx'^i = g'_{ji} \, dx'^j \, dx'^i$$

Now:

$$a^i b_i = g^{ij} a_j b_i = g_{ij} a^i b^j = a_i b^i$$

If $g^{ij} = \mathbb{I}$ then there is no difference between covariant and contravariant components:

$$a^i = g^{ij} a_j = \delta^{ij} a_j = a_i \quad \text{no difference.}$$

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Levi-Civita tensor.

In 3D:

$$\varepsilon^{ijk} = \varepsilon_{ijk} = \begin{cases} 1 & \text{if } i \neq j \neq k \longrightarrow \text{cyclic order } 1,2,3, \text{ etc.} \\ -1 & \text{if } i \neq j \neq k \longrightarrow \text{not cyclic order } 1,3,2 \\ 0 & \text{otherwise.} \end{cases}$$

Rank: 3 (its rank equals N the space dimension).

Antisymmetric in all pair of indices.

It has $3^3 = 27$ components.

It has 6 non-zero elements: 123 231 312
132 321 213

It has 1 independent component.

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Properties:

$$\varepsilon_{pqr} = \hat{x}^p \cdot (\hat{x}^q \times \hat{x}^r)$$

$\hat{x}^i$: coordinate unit vectors.

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \equiv a_{1i}\, a_{2j}\, a_{3k}\, \varepsilon_{ijk}$$

(see it expanding this).

Also

$$\det A \; \varepsilon_{\alpha\beta\gamma} = a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, \varepsilon_{ijk}$$

$$\| $$

$$a_{1i}\, a_{2j}\, a_{3k}\, \varepsilon_{ijk}\, \varepsilon_{\alpha\beta\gamma}$$

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Cross Product

$$C_i = \varepsilon_{ijk} A^j B^k \equiv \bar{C} = \bar{A} \times \bar{B}$$

$$C_1 = \underbrace{\varepsilon_{123}}_{1} A^2 B^3 + \underbrace{\varepsilon_{132}}_{-1} A^3 B^2 =$$

$$= A^y B^z - A^z B^y = C_x$$

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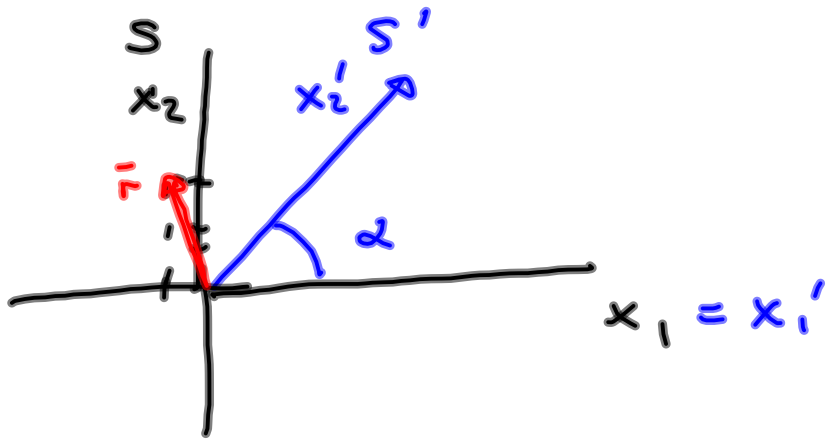
Tips for the test:

$\hat{e}_1 = (1,0)$
$\hat{e}_2 = (0,1)$

$\alpha = 45°$



$\bar{r} = (-1, 3)$ in S

$\phi(x_1, x_2) = x_1^2 + x_2^2$

$\bar{r} = -\hat{e}_1 + 3\hat{e}_2$

We found that

In S'

$$\begin{cases} x'^1 = x_1 - \cot\alpha\, x_2 = x_1 - \cot 45°\, x_2 = x_1 - x_2 \\ x'^2 = x_2 \operatorname{cosec}\alpha = x_2 \operatorname{cosec} 45° = \sqrt{2}\, x_2 \end{cases}$$

a) obtain $\phi(\bar{r})$ in S: $\phi(x_1 = -1, x_2 = 3) = (-1)^2 + 3^2 = 10$.

b) Find the contravariant coordinates of $\bar{r}'$ in S':

$$x'^1 = x_1 - x_2 = -1 - 3 = -4$$

$$x'^2 = \sqrt{2}\, x_2 = 3\sqrt{2}$$

$$\bar{r}' = -4\, \hat{e}'_1 + 3\sqrt{2}\, \hat{e}'_2$$

$$\hat{e}'_1 = (1, 0)$$

$$\hat{e}'_2 = (\cos\alpha, \sin\alpha) = \frac{\sqrt{2}}{2}(1, 1)$$

c) Calculate $\phi'(x'^1, x'^2)$ in S':

$$\phi(x_1, x_2) = x_1^2 + x_2^2 = (\underbrace{x^{1\prime} + \frac{\sqrt{2}}{2}\, x^{2\prime}}_{x_1})^2 + (\underbrace{\frac{\sqrt{2}}{2}\, x^{2\prime}}_{x_2})^2 =$$

$$= (x'^1)^2 + (x'^2)^2 + \sqrt{2}\, x'^1 x'^2$$

d) Evaluate $\phi'(\bar{r}')$ at $\bar{r}' = (\underbrace{-4}_{x^{1\prime}}, \underbrace{3\sqrt{2}}_{x^{2\prime}})$

$$\phi'(-4, 3\sqrt{2}) = 16 + 18 + \sqrt{2}(-4)(3\sqrt{2}) = 10$$

e) Calculate $\partial_i \phi(x_1, x_2)$ in S:

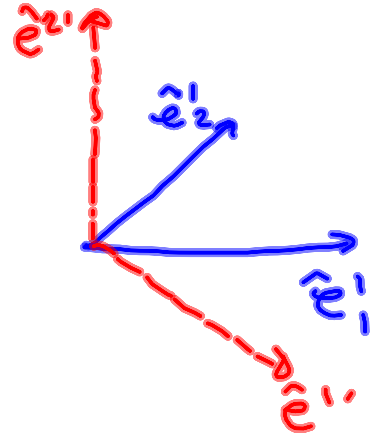
$$\phi(x_1, x_2) = x_1^2 + x_2^2$$

$$\partial_i \phi(x_1, x_2) = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\right) = (2x_1, 2x_2)$$

$$\partial_i \phi(x_1, x_2) = B_i = (2x_1, 2x_2)$$

$$\bar{B} = 2x_1 \hat{e}_1 + 2x_2 \hat{e}_2$$

f) Evaluate $\partial_i \phi(x_1, x_2)$ at $\bar{r}$ in S:



$\partial_i \phi(-1, 3) = (-2, 6)$ vector.

g) Calculate $\partial'_i \phi'(x'^1, x'^2)$ in S':

$$\phi'(x'^1, x'^2) = (x'^1)^2 + (x'^2)^2 + \sqrt{2}\, x'^1 x'^2$$

$$\partial'_i \phi' = \left(\frac{\partial \phi'}{\partial x'^1}, \frac{\partial \phi'}{\partial x'^2} \right) = \left(2x'^1 + \sqrt{2}\, x'^2, 2x'^2 + \sqrt{2}\, x'^1 \right)$$

$$= B'_i$$

$$\bar{B}' = (2x'^1 + \sqrt{2}\, x'^2)\, \hat{e}'^1 + (2x'^2 + \sqrt{2}\, x'^1)\, \hat{e}'^2$$

Covariant components.

h) Evaluate $\partial'_i \phi'$ at $\bar{r}'$ in S': $\bar{r}' = (-4, 3\sqrt{2})$

$B'_i = (-2, 2\sqrt{2})$

$\bar{B}' = -2\,\hat{e}'^1 + 2\sqrt{2}\,\hat{e}'^2$

i) $|\bar{B}|(\bar{r})$ in S: $\bar{B} = (-2, 6)$

$|\bar{B}| = (\bar{B}\cdot\bar{B})^{1/2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$

j) $|\bar{B}'|(\bar{r}')$ in S':

$|\bar{B}'| = (B'_i \cdot B'^i)^{1/2} =$

Let's find B'^{i}

B'^{i} as a vector is

$$\bar{B}' = a\,\underset{\underset{(1,0)}{\shortparallel}}{\hat{e}'_1} + b\,\underset{\underset{\frac{\sqrt{2}}{2}(1,1)}{\shortparallel}}{\hat{e}'_2} = \bar{B} = -2\,\underset{\underset{(1,0)}{\downarrow}}{\hat{e}_1} + 6\,\underset{\underset{(0,1)}{\downarrow}}{\hat{e}_2}$$

$a = -8 \qquad b = 6\sqrt{2}$

$\bar{B}'^{i} = (-8, 6\sqrt{2}) \qquad \bar{B}'_i = (-2, 2\sqrt{2})$

Then

$$|\bar{B}'| = \sqrt{B'_i B'^{i}} = \sqrt{16+24} = \sqrt{40} = \boxed{2\sqrt{10}} \equiv |\bar{B}|$$

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