Midterm Exam

P571
October 2, 2012

SHOW ALL WORK TO GET FULL CREDIT!

PART I: **DO IT IN CLASS** Turn your work in before leaving. Take the printed copy of the test home.

PART II: Take the test home and bring **ALL** the questions solved on Tuesday October 9. Your grade for the test will be the **sum of the two** parts. Each question is worth 4 points. A perfect score is worth 108 points as a result of 24 points to be earned in class and 84 points to be earned at home. If you are 100% sure about the work you did in class, you do not need to redo it at home. In that case the points obtained in class will be counted twice.

PART I

:

Consider a system of coordinates S' in two-dimensional space with covariant basis vectors $\hat{\mathbf{e}'}_i$. Vector $\hat{\mathbf{e}'}_1$ makes an angle $\beta = -30°$ with $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}'}_2 = \hat{\mathbf{e}}_2$ where $\hat{\mathbf{e}}_i$ are the basis vectors of the cartesian system that we'll call S (see figure).

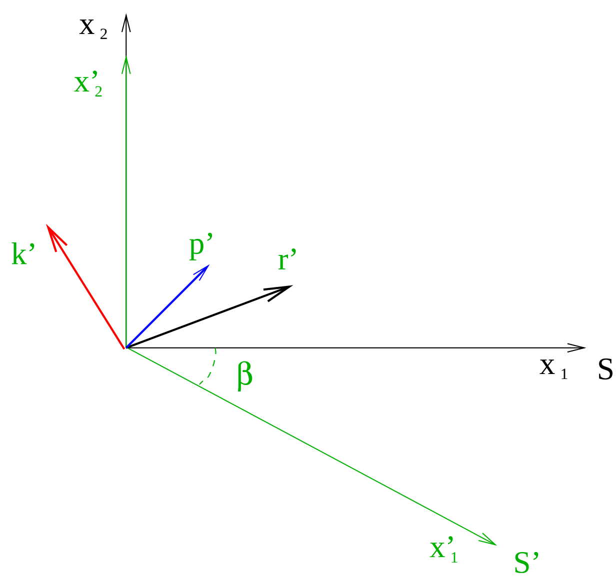

FIG. 1: The figure is approximated, i.e., not to scale which means that you cannot "read" the answers from the graph. Hint: making your own figure may be helpful.

a) Provide expressions for the two versors $\hat{\mathbf{e}'}_i$ in terms of the orthogonal basis versors $\hat{\mathbf{e}}_i$.

b) Calculate the vectors $\hat{\mathbf{e}'}^i$ that form the contravariant basis of S'. Provide expressions for them in terms of the orthogonal basis versors $\hat{\mathbf{e}}_i$.

c) Now consider a generic vector r'^i with contravariant components x'^i in S' and provide its components x^i in the orthogonal system S (see Fig.1).

d) Provide the covariant components x'_i in S' of the generic vector r'^i considered in part (c).

Now consider the vectors $\mathbf{p}'$ and $\mathbf{k}'$ (see Fig. 1)given by

$$\mathbf{p}' = 2\hat{\mathbf{e}'}_1 + 3\hat{\mathbf{e}'}_2,$$

and

$$\mathbf{k}' = -2\hat{\mathbf{e}'}_1 + 2\hat{\mathbf{e}'}_2.$$

Hint: remember that a vector r'^i with contravariant components x'^i can be expressed as $\mathbf{r}' = x'^1\hat{\mathbf{e}'}_1 + x'^2\hat{\mathbf{e}'}_2$.

e) Calculate the components of $\mathbf{p}'$ and $\mathbf{k}'$ in the orthogonal system S.

f) Calculate the covariant components of $\mathbf{p}'$ and $\mathbf{k}'$ in system S'.

STOP HERE!!!!: Hand your work to the proctor before leaving and take home the printed copy of the test.

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g) Calculate the angle between $\mathbf{p}$ and $\mathbf{k}$ in S.

h) Calculate the angle between $\mathbf{p}'$ and $\mathbf{k}'$ in S'. Compare your result with the result you obtained in (g) and comment.

Now consider the function $\Phi(x_1, x_2) = x_1^2 - x_2$ in system S (remember that $x_i = x^i$ in S and I'm putting the indices down so that the exponent for x_1 is clear).

i) Is $\Phi(x_1, x_2)$ a tensor? If you say yes, provide its rank.

j) Provide an expression for $\Phi'(x'^1, x'^2)$ in S'. Hint: the results found in part (c) could be useful.

k) Evaluate $\Phi'(x'^1, x'^2)$ at $\mathbf{p}'$.

l) Evaluate $\Phi(x^1, x^2)$ at $\mathbf{p}$. Compare with the result you obtained in (k).

m) Calculate $\partial_i \Phi(x^1, x^2)$, i.e., the gradient of the function Φ in S.

n) Is $\partial_i \Phi(x^1, x^2)$ a tensor? If you say yes, provide its rank.

o) Calculate $\partial'_i \Phi'(x'^1, x'^2)$, i.e., the gradient of the function Φ' in S'.

p) Evaluate $\partial_i \Phi(x^1, x^2)$ at $\mathbf{p}$.

q) Evaluate $\partial'_i \Phi(x'^1, x'^2)$ at $\mathbf{p}'$. Compare with the result you obtained in (p).

r) Calculate the norm of $\partial_i \Phi(x^1, x^2)$ at $\mathbf{p}$.

s) Calculate the norm of $\partial'_i \Phi(x'^1, x'^2)$ at $\mathbf{p}'$. Compare with the result that you obtained in (r).

t) Calculate the direct (or outer) product between $\mathbf{p}$ and $\mathbf{k}$ in S and call the result T. Calculate the trace of T. Hint: Provide the components of the rank 2 tensor $T^{ij} = p^i k^j$.

u) Provide expressions for T'^{ij}, T'_{ij}, $T'^{i}{}_{j}$, and T'^{j}_{i}, i.e. T in system S'. Which of the four tensors have the same trace as T? Why?