Midterm Exam

P571

October 2, 2012

SOLUTION:

a) We need to express $\hat{\mathbf{e}}'_i$ in cartesian coordinates.

$$\hat{\mathbf{e}}'_1 = (\cos 30^\circ, -\sin 30^\circ) = (\frac{\sqrt{3}}{2}, -\frac{1}{2}), \tag{1}$$

and

$$\hat{\mathbf{e}}'_2 = (0, 1). \tag{2}$$

b) In order to find the vectors $\hat{\mathbf{e}}'^i$ that form the contravariant basis we have to use that

$$\hat{\mathbf{e}}'^i . \hat{\mathbf{e}}'_j = \delta^i{}_j. \tag{3}$$

Let's propose that $\hat{\mathbf{e}}'^1 = (a, b)$ and $\hat{\mathbf{e}}'^2 = (c, d)$. Then using the values obtained in Eq.(1) and (2) in Eq.(3) we can solve for a, b, c, and d.

$$0 = \hat{\mathbf{e}}'^1 . \hat{\mathbf{e}}'_2 = b,$$

then $b = 0$,

$$1 = \hat{\mathbf{e}}'^1 . \hat{\mathbf{e}}'_1 = \frac{a\sqrt{3}}{2},$$

then $a = \frac{2\sqrt{3}}{3}$. Then

$$\hat{\mathbf{e}}'^1 = (\frac{2\sqrt{3}}{3}, 0), \tag{4}$$

and

$$1 = \hat{\mathbf{e}}'^2 . \hat{\mathbf{e}}'_2 = d,$$

then $d = 1$,

$$0 = \hat{\mathbf{e}}'^2 . \hat{\mathbf{e}}'_1 = \frac{c\sqrt{3}}{2} - \frac{1}{2},$$

then $c = \frac{\sqrt{3}}{3}$. Then

$$\hat{\mathbf{e}}'^2 = (\frac{\sqrt{3}}{3}, 1). \tag{5}$$

c) In S' a generic vector r'^i has coordinates x'^i and in S a generic vector r^i has coordinates x^i. From the figure we see that

$$x^1 = x'^1 \cos\beta = x'^1 \cos 30^\circ = x'^1 \frac{\sqrt{3}}{2}, \tag{6}$$

and

$$x^2 = x'^2 - x'^1 \sin\beta = -x'^1 \sin 30^\circ + x'^2 = -x'^1 \frac{1}{2} + x'^2. \quad (7)$$

d) The easiest way to do this is remembering that a generic vector $\mathbf{r}'$ can be written in terms of its covariant components in the contravariant basis as:

$$\mathbf{r}' = x'_1 \hat{\mathbf{e}}'^1 + x'_2 \hat{\mathbf{e}}'^2, \quad (8)$$

and that it also can be written in terms of its contravariant components in the covariant basis then we have that

$$\mathbf{r}' = x'_1 \hat{\mathbf{e}}'^1 + x'_2 \hat{\mathbf{e}}'^2 = x'^1 \hat{\mathbf{e}}'_1 + x'^2 \hat{\mathbf{e}'}_2. \quad (9)$$

Then using the cartesian expressions for $\hat{\mathbf{e}}'_i$ and $\hat{\mathbf{e}}'^i$ found in parts (a) and (b) we obtain:

$$x'_1(\frac{2\sqrt{3}}{3}, 0) + x'_2(\frac{\sqrt{3}}{3}, 1) = x'^1(\frac{\sqrt{3}}{2}, -\frac{1}{2}) + x'^2(0, 1),$$

which gives us two equations:

$$x'_1 \frac{2\sqrt{3}}{3} + x'_2 \frac{\sqrt{3}}{3} = x'^1 \frac{\sqrt{3}}{2} \quad (10)$$

and

$$x'_2 = -\frac{x'^1}{2} + x'^2. \quad (11)$$

Replacing (11) in (10) we obtain:

$$x'_1 = x'^1 - x'^2 \frac{1}{2}. \quad (12)$$

e) Now we simply have to use Eq.(6) and (7). For $\mathbf{p}$ I obtain:

$$p^i = (2\frac{\sqrt{3}}{2}, -2\frac{1}{2} + 3) = (\sqrt{3}, 2), \quad (13)$$

and for $\mathbf{k}$

$$k^i = (-2\frac{\sqrt{3}}{2}, 2\frac{1}{2} + 2) = (-\sqrt{3}, 3). \quad (14)$$

f) Now we just need to plug the contravariant components of vectors p in Eqs.(11) and (12):

$$p'_i = (2 - \frac{3}{2}, -\frac{2}{2} + 3) = (\frac{1}{2}, 2), \quad (15)$$

and for vector k:

$$k'_i = (-2 - \frac{2}{2}, \frac{2}{2} + 2) = (-3, 3). \quad (16)$$

g) We know that in S

$$\mathbf{p}.\mathbf{k} = p_i k^i = p_i k_i = pk \cos\alpha, \quad (17)$$

Then

$$\alpha = \cos^{-1}\frac{\mathbf{p}.\mathbf{k}}{pk}. \tag{18}$$

Let's calculate the absolute values:

$$p = \sqrt{3+4} = \sqrt{7}, \tag{19}$$

and

$$k = \sqrt{3+9} = 2\sqrt{3}. \tag{20}$$

Then

$$\alpha = \cos^{-1}(\frac{-3+6}{2\sqrt{21}}) = \cos^{-1}(\frac{3}{2\sqrt{21}}) = 70.89^o \tag{21}$$

h) Now I have to repeat the same calculation but in S' where

$$\mathbf{p}'.\mathbf{k}' = p'_i k'^i = p'k'\cos\alpha', \tag{22}$$

Then

$$\alpha' = \cos^{-1}\frac{p'_i k'^i}{p'k'}. \tag{23}$$

Let's calculate the absolute values. Now I need to use the expression given for the contravariant components and the covariant components found in part (f):

$$p' = \sqrt{\frac{1}{2}2 + 2\times 3} = \sqrt{7}, \tag{24}$$

which equals p as expected since the magnitude of a vector is a scalar and

$$k' = \sqrt{(-3)(-2)+3\times 2} = 2\sqrt{3}, \tag{25}$$

which, as expected, equals the magnitude k. Then

$$\alpha' = \cos^{-1}(\frac{-1+4}{2\sqrt{21}}) = \cos^{-1}(\frac{3}{2\sqrt{21}}) = 70.89^o. \tag{26}$$

We see that $\alpha' = \alpha$ since the angle between the two vectors is a scalar.

i) Since at each point (x_1, x_2) the function Φ provides a scalar value we see that Φ is a scalar function, i.e., a tensor of rank 0.

j) All we need to do is to write Φ in terms of the contravariant variables in S'. Thus, using Eq.(6) and (7) we obtain

$$\Phi'(x'^1, x'^2) = (x'^1\frac{\sqrt{3}}{2})^2 - (-x'^1\frac{1}{2} + x'^2) = \frac{3}{4}(x'^1)^2 + \frac{x'^1}{2} - x'^2. \tag{27}$$

k) Using the contravarian coordinates of p'^i provided I obtain:

$$\Phi'(2,3) = \frac{3}{4}(2)^2 + \frac{2}{2} - 3 = 3 + 1 - 3 = 1. \tag{28}$$

l) Using the cartesian coordinates of p^i obtained in (e):

$$\Phi(\sqrt{3}, 2) = 3 - 2 = 1, \tag{29}$$

which is the same result obtained in (k) because the function is a scalar.

m) Now we need to calculate $\partial_i \Phi$.

$$\partial_i \Phi = (\frac{\partial \Phi}{\partial x_1}, \frac{\partial \Phi}{\partial x_2}) = (2x_1, -1) \tag{30}$$

n) The gradient of the scalar function Φ is a vector and thus, it is a tensor of rank 1.

o) Now we need to calculate $\partial'_i \Phi'$, i.e., the gradient of the function in S'. The easiest way is to use the expression for the transformed function that we obtained in (j), then:

$$\partial'_i \Phi' = (\frac{\partial \Phi'}{\partial x'^1}, \frac{\partial \Phi'}{\partial x'^2}) = (\frac{3}{2}x'^1 + \frac{1}{2}, -1), \tag{31}$$

Notice that the coordinates obtained in Eq.(31) covariant in S', i.e., it means that

$$\partial'_i \Phi' = (\frac{3}{2}x'^1 + \frac{1}{2})\hat{\mathbf{e}}'^1 - \hat{\mathbf{e}}'^2. \tag{32}$$

p) Using the result obtained in (m) and the result obtained in (e) for the coordinates of p^i in S we obtain

$$\partial_i \Phi(\sqrt{3}, 2) = (2\sqrt{3}, -1). \tag{33}$$

q) Using the result obtained in (o) and the coordinates of p'^i in S' given in the problem we obtain

$$\partial'_i \Phi'(2,3) = (\frac{3}{2}2 + \frac{1}{2})\hat{\mathbf{e}}'^1 - 3\hat{\mathbf{e}}'^2 = (\frac{7}{2})\hat{\mathbf{e}}'^1 - \hat{\mathbf{e}}'^2. \tag{34}$$

r) The norm of $\partial_i \Phi(\sqrt{3}, 2)$ in S is obtained as

$$\sqrt{\partial_i \Phi(\sqrt{3}, 2)\partial_i \Phi(\sqrt{3}, 2)} = \sqrt{12 + 1} = \sqrt{13}. \tag{35}$$

s) The norm of $\partial'_i \Phi'(2,3)$ in S' is obtained as

$$\sqrt{\partial'_i \Phi'(2,3)\partial'^i \Phi'(2,3)}. \tag{36}$$

I need to calculate the contravariant coordinates of $\partial'^i\Phi(\sqrt{3},2)$ so that we can write

$$\partial'^i\Phi'(x'^1=2,x'^2=3)=a\hat{\mathbf{e}}'_1+b\hat{\mathbf{e}}'_2. \tag{37}$$

We know that as a vector

$$\partial'^i\Phi'(x'^1,x'^2)=\partial^i\Phi(x^1,x^2),$$

which means that

$$a\hat{\mathbf{e}}'_1+b\hat{\mathbf{e}}'_2=2\sqrt{3}\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2. \tag{38}$$

Using the expression for $\hat{\mathbf{e}}'_i$ in the cartesian system S that we found in part (a) Eq.(38) becomes:

$$a(\frac{\sqrt{3}}{2},-\frac{1}{2})+b(0,1)=(2\sqrt{3},-1). \tag{39}$$

Then

$$(a\frac{\sqrt{3}}{2},-\frac{a}{2})+b)=(2\sqrt{3},-1), \tag{40}$$

and we obtain that $a=4$ and $b=1$ then,

$$\partial'^i\Phi'(2,3)=4\hat{\mathbf{e}}'_1+\hat{\mathbf{e}}'_2. \tag{41}$$

Now replacing in Eq.(36) we obtain:

$$\sqrt{\partial'_i\Phi'(2,3)\partial'^i\Phi'(2,3)}=\sqrt{\frac{7}{2}4+(-1)(1)}=\sqrt{14-1}=\sqrt{13}. \tag{42}$$

We see that the result is the same as the one obtained in (r) as expected since the norm of a vector is a scalar.

t) Now we need to construct the tensor $T^{ij}=p^ik^j$ in S. Thus, we use the coordinates for $p^i=(\sqrt{3},2)$ and $k^j=(-\sqrt{3},3)$ obtained in (d):

$$T^{ij}=\begin{pmatrix} p^1k^1 & p^1k^2 \\ p^2k^1 & p^2k^2 \end{pmatrix}=\begin{pmatrix} -3 & 3\sqrt{3} \\ -2\sqrt{3} & 6 \end{pmatrix}. \tag{43}$$

The trace of T is just -3+6=3.

u) The easiest way to do this is to use the covariant and contravariant coordinates of p'^i and k'^j given in the problem and calculated in part (f) which means that $p'^i=(2,3)$, $k'^j=(-2,2)$, $p'_i=(\frac{1}{2},2)$ and $k'_j=(-3,3)$ then:

$$T'^{ij}=\begin{pmatrix} p'^1k'^1 & p'^1k'^2 \\ p'^2k'^1 & p'^2k'^2 \end{pmatrix}=\begin{pmatrix} -4 & 4 \\ -6 & 6 \end{pmatrix}. \tag{44}$$

$$T'_{ij}=\begin{pmatrix} p'_1k'_1 & p'_1k'_2 \\ p'_2k'_1 & p'_2k'_2 \end{pmatrix}=\begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ -6 & 6 \end{pmatrix}. \tag{45}$$

$$T'^j_i=\begin{pmatrix} p'_1k'^1 & p'_1k'^2 \\ p'_2k'^1 & p'_2k'^2 \end{pmatrix}=\begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix}. \tag{46}$$

$$T'^i{}_j=\begin{pmatrix} p'^1k'_1 & p'^1k'_2 \\ p'^2k'_1 & p'^2k'_2 \end{pmatrix}=\begin{pmatrix} -6 & 6 \\ -9 & 9 \end{pmatrix}. \tag{47}$$

We see that the traces of T'^{ij}, T'_{ij}, T'^j_i and $T'^i{}_j$ are 2, 9/2, 3, and 3. Only the traces of the mixed forms of the tensor have the same value as the trace of T in S. This is because these are the only traces that are tensors obtained from the contraction of the indices of T as T'^i_i and $T'^i{}_i$. The traces of T'^{ij}, T'_{ij} are not tensors.