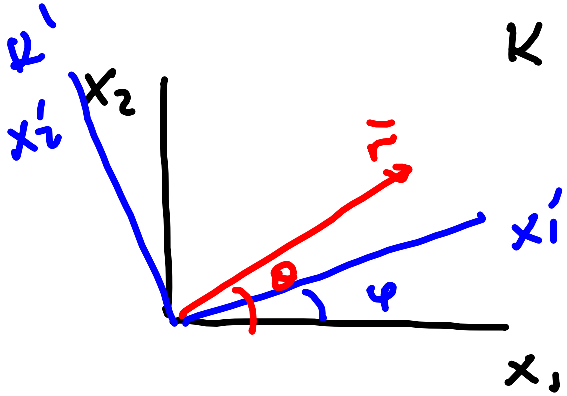
Last Time: 8/27



$$x_1' = x_1 \cos\varphi + x_2 \sin\varphi$$

$$x_2' = -x_1 \sin\varphi + x_2 \cos\varphi$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x'_i = M_{ij} x_j \qquad M_{ij} = \frac{\partial x'_i}{\partial x_j}$$

x^i represent a vector in K eg

$$\boxed{x'^i = \frac{\partial x'^i}{\partial x_j} x^j}$$

$$\frac{\partial x'_1}{\partial x_2} = \sin\varphi \qquad \frac{\partial x'_2}{\partial x_2} = \cos\varphi$$

In N dimensions a vector is given by:

$$r_i = (x_1, x_2, \ldots, x_N) \quad \text{in system } K.$$

In K':

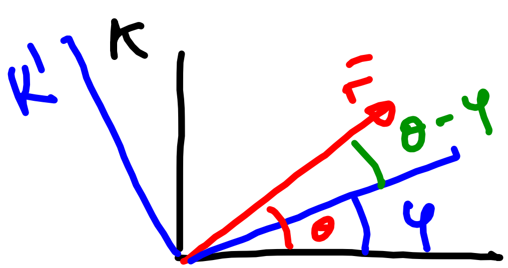
$$x'_i = \sum_{j=1}^{N} M_{ij} x_j = \sum_{j=1}^{N} \frac{\partial x'_i}{\partial x_j} x_j \equiv$$

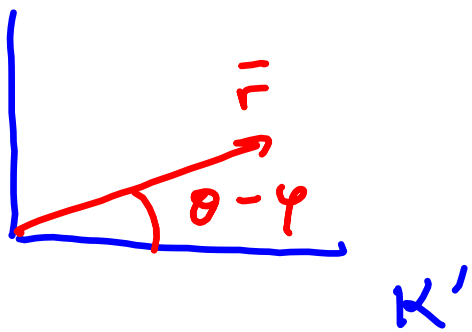
$$\equiv \frac{\partial x'^i}{\partial x^j} x^j$$

Einstein's notation.

Contravariant Vector

$\bar{r}$ is the prototype contravariant tensor because if K is rotated to K' counterclockwise, the coordinates x'^i of $\bar{r}$ in K' are the same as the coordinates of a vector rotated clockwise in K.



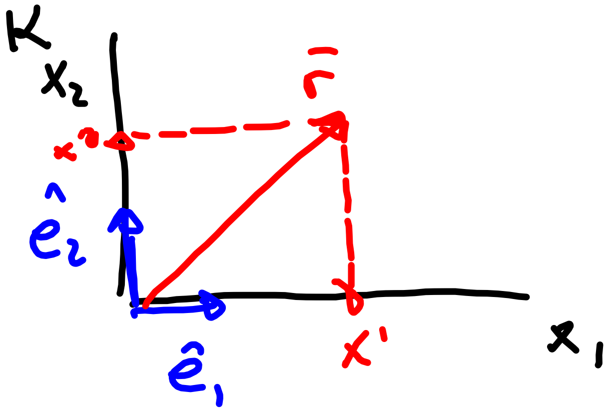


A contravariant vector's coordinates are represented as a column matrix since:

$$x'^i = M_{ij} x^j \implies \begin{pmatrix} x'^1 \\ \vdots \\ x'^N \end{pmatrix} = \begin{pmatrix} M_{11} & \cdots & M_{1N} \\ \vdots & & \vdots \\ M_{N1} & \cdots & M_{NN} \end{pmatrix} \begin{pmatrix} x^1 \\ \vdots \\ x^N \end{pmatrix}$$

Covariant vectors

In K:



$$\bar{r} = x^1\hat{e}_1 + x^2\hat{e}_2 = \sum_{i=1}^{2} x^i\hat{e}_i \equiv$$

$$\equiv x^i\hat{e}_i \quad (1)$$

Einstein's notation

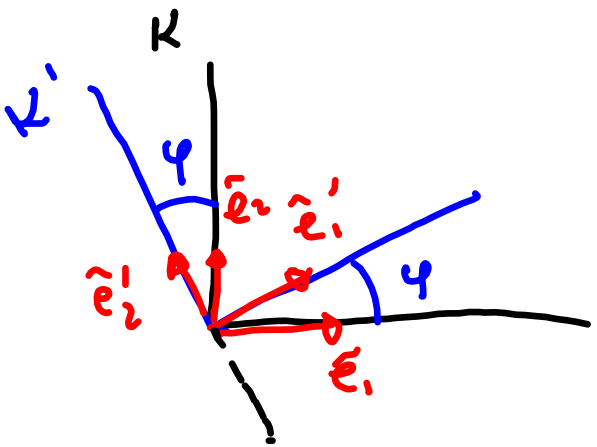
$\hat{e}_i$: versor (length 1)

As matrices if x^i is a column matrix then $\hat{e}_i$ has to be a row matrix to represent eq. (1).

$$\bar{r} = (\hat{e}_1, \hat{e}_2)\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = x^1\hat{e}_1 + x^2\hat{e}_2$$

The basis vectors $\hat{e}_i$ are covariant because if K is rotated by φ counterclockwise to become K' then $\hat{e}_i$ rotate counterclockwise as well.

Let's see how the $\hat{e}_i$ vectors transform from K to K'



$$\hat{e}'_1 = \hat{e}_1 \cos\varphi + \hat{e}_2 \sin\varphi$$

$$\hat{e}'_2 = -\hat{e}_1 \sin\varphi + \hat{e}_2 \cos\varphi$$

$$(\hat{e}'_1, \hat{e}'_2) = (\hat{e}_1, \hat{e}_2) \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{A^j{}_i} =$$

$$= \hat{e}_j A^j{}_i$$

We saw that

$$M^i{}_j = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$ then we see that

$$AM = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then $A = M^{-1}$

Covariant vectors transform like the inverse of contravariant vectors.

$$(AM)^i{}_j = \sum_{k=1}^{N} A^i{}_k M^k{}_j = \delta^i{}_j$$

Kronecker's tensor

$$\begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

Then we can see that

$$M^i{}_j = \frac{\partial x'^i}{\partial x^j}$$

and since A is the inverse transformation

$$A^i{}_j = \frac{\partial x^i}{\partial x'^j}$$

check using that

$$x^1 = x'^1 \cos\varphi - x'^2 \sin\varphi$$
$$x^2 = x'^1 \sin\varphi + x'^2 \cos\varphi$$

Then

$$x'^i = \frac{\partial x'^i}{\partial x^j} x^j \equiv M^i{}_j \, x^j \text{ " up}$$

contravariant

$$\hat{e}'_j = \frac{\partial x^i}{\partial x'^j} \hat{e}_i \equiv A^i{}_j \, \hat{e}_i \text{ " down}$$

covariant

Notation :

contravariant : index up because x' is upstairs in $\frac{\partial x'^i}{\partial x^j}$

covariant : index down because x' is downstairs in $\frac{\partial x^i}{\partial x'^j}$

Components of a vector

A contravariant vector such as $\bar{r}$ can be written in terms of its contravariant components x^i or covariant components x_i.

(The same is true for a covariant vector).

This is important to perform scalar products.

Consider:

$$|\bar{r}|^2 = \bar{r}\cdot\bar{r} = \sum_{i=1}^{N} x_i x_i \equiv \sum_{i=1}^{N} x_i x^i$$

As matrices you need to use

$$x_i = (x_1, x_2, \dots, x_N) \quad \text{and} \quad x^i = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

then

$$x_i x^i = (x_1 \dots x_N)\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = x_1^2 + x_2^2 + \dots + x_N^2 = \bar{r}\cdot\bar{r}$$

In cartesian coordinates $x_i = x^i$ (not true in general).

Then we obtain

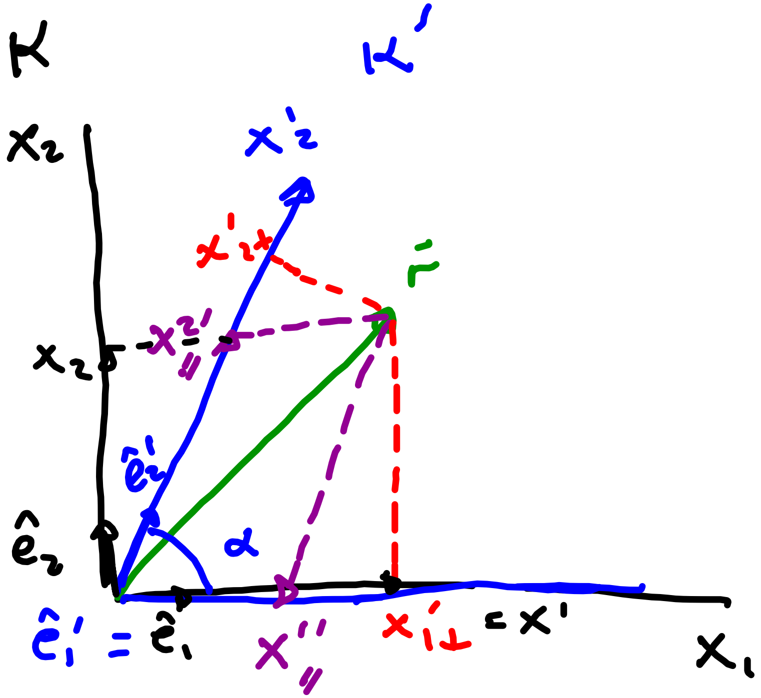
$$r^{i'} = \frac{\partial x'^{i}}{\partial x^{j}} r^{j} \quad \text{or} \quad x^{i'} = \frac{\partial x'^{i}}{\partial x^{j}} x^{j}$$

but

$$r'_{i} = \frac{\partial x^{j}}{\partial x'^{i}} r_{j} \quad \text{or} \quad x'_{i} = \frac{\partial x^{j}}{\partial x'^{i}} x_{j}$$

inverse transformation

Non-trivial Example: oblique system



In K: $\bar{r} = x^i = (x^1, x^2)$

In K' we have

$\bar{r} = (x^1_{\parallel}, x^2_{\parallel})$ or

$\bar{r} = (x_{1\perp}, x_{2\perp})$

$\bar{r} = x^j \hat{e}_j$

$\bar{r} = x'^j \hat{e}'_j$

contravariant components

Is $x'^j = x^j_{\parallel}$ or $x_{\perp j}$?

$$X_1 = x^{1'}_{\parallel} + x^{2'}_{\parallel} \cos\alpha$$

$$X_2 = x^{2'}_{\parallel} \sin\alpha$$