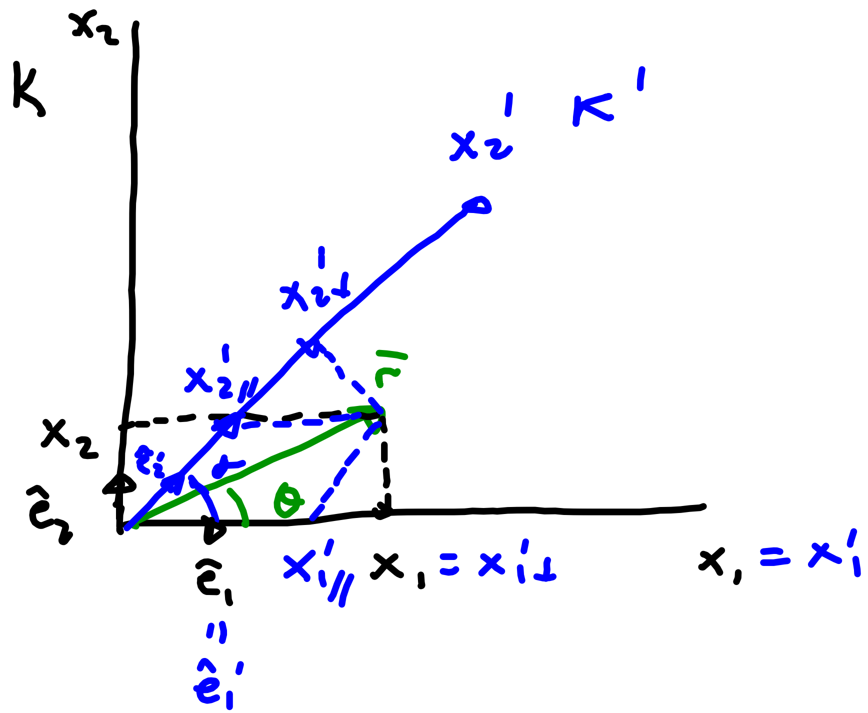
8/29



In K $\vec{r} = x^j \hat{e}_j$

In K' $\vec{r} = x'^j \hat{e}_j'$

Parallel components:

$$\begin{cases} x_1 = x'_{1\parallel} + x'_{2\parallel} \cos\alpha \\ x_2 = x'_{2\parallel} \sin\alpha \quad (1) \end{cases}$$

$$\begin{cases} x'_{1\parallel} = x_1 - x'_{2\parallel}\cos\alpha \overset{(1)}{=} \\ \qquad = x_1 - x_2 \operatorname{cotan}\alpha \\ x'_{2\parallel} = x_2 \csc\alpha \quad (2) \end{cases}$$

Perpendicular components

$$\begin{cases} x'_{1\perp} = x_1 \\ x'_{2\perp} = x'_{2\parallel} + x'_{1\parallel}\cos\alpha \overset{(2)}{=} \\ = x_2 \csc\alpha + x_1\cos\alpha - \\ \quad - x_2 \frac{\cos^2\alpha}{\sin\alpha} = \\ = x_1 \cos\alpha + x_2 \underbrace{\left(\frac{1}{\sin\alpha} - \frac{\cos^2\alpha}{\sin\alpha}\right)}_{\sin\alpha} = \\ = x_1\cos\alpha + x_2 \sin\alpha \end{cases}$$

$$\begin{cases} x'_{1\perp} = x_1 \\ x'_{2\perp} = \cos\alpha\, x_1 + \sin\alpha\, x_2 \end{cases}$$

$$\begin{cases} x_1 = x'_{1\perp} \\ x_2 = -\operatorname{cotan}\alpha\, x'_{1\perp} + \csc x'_{2\perp} \end{cases}$$

How do $\{\hat{e}_i\}$ transform? $\hat{e}_i$ are covariant

$$\hat{e}_1' = \hat{e}_1$$

$$\hat{e}_2' = \cos\alpha\, \hat{e}_1 + \sin\alpha\, \hat{e}_2$$

In matrix form

$$(\hat{e}_1', \hat{e}_2') = (\hat{e}_1, \hat{e}_2) \underbrace{\begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix}}_{A^i{}_j}$$

we know that

$$A^i{}_j = \frac{\partial x^i}{\partial x'^j} \equiv \frac{\partial x^i}{\partial x'^j}$$

→ contravariant

We see that $x'^i_{\parallel} = x'^i$ in the oblique system.

Then $$M^i{}_j = \frac{\partial x'^i}{\partial x^j} = \frac{\partial x'^i_{\parallel}}{\partial x^j} = \begin{pmatrix} 1 & -\cot\alpha \\ 0 & \csc\alpha \end{pmatrix}$$

then: $$x'_{i\parallel} = M^i{}_j\, x_j \quad \text{or} \quad \begin{pmatrix} x'^1_{\parallel} \\ x^{2\prime}_{\parallel} \end{pmatrix} = \underset{M}{\begin{pmatrix} & \\ & \end{pmatrix}} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

and

$$x'_{i\perp} = A^j{}_i\, x_j \quad \text{or} \quad (x_1'{}_\perp, x_2'{}_\perp) = (x_1, x_2) \underset{A}{\begin{pmatrix} & \\ & \end{pmatrix}}$$

Notice that $AM = \mathbb{I} \Rightarrow A = M^{-1}$

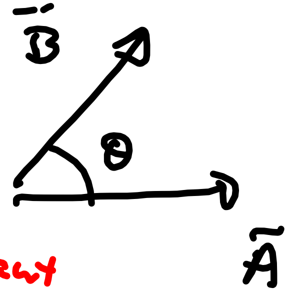
or $A^i{}_j M^j{}_k = \delta^i{}_k$

Vector operations

Scalar or dot product (particular case of "contraction").

You know that

$$\bar{A}\cdot\bar{B} = \sum_{i=1}^{N} A_i B_i = \sum_{i=1}^{N} B_i A_i = \bar{B}\cdot\bar{A}$$



As matrices:

contravariant

$$\bar{A}\cdot\bar{B} = (a_1,\dots,a_N)\begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} = \sum_{i=1}^{N} a_i b^i \equiv a_i b^i$$

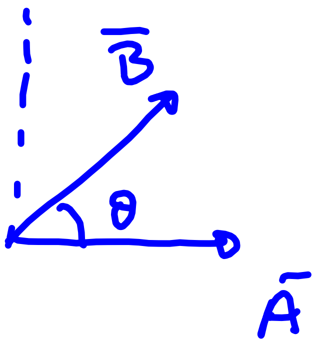
covariant

Norm or length of a vector:

$$\bar{A}\cdot\bar{A} = a_i a^i = \sum_{i=1}^{N} a_i^2 = |A|^2$$

scalar or tensor of rank 0.

$$|A| = (\bar{A}\cdot\bar{A})^{1/2} \equiv A \quad \text{scalar length.}$$



$$\bar{A}\cdot\bar{B} = A\,\underbrace{B_{\parallel}}_{B\cos\theta} + A_{\perp}\underbrace{B_{\perp}}_{0} = AB\cos\theta$$

$B_{\parallel}$: along $\bar{A}$ direction

$$\cos\theta = \frac{\bar{A}\cdot\bar{B}}{AB} = \frac{a_i b^i}{(a_k a^k)^{1/2}(b_j b^j)^{1/2}}$$

Scalar product in oblique system:

In K: $|\bar{r}|^2 = x_i x^i = x_1^2 + x_2^2$

In K' (oblique):

$$|\bar{r}'|^2 = x_1' x^{1\prime} + x_2' x^{2\prime} =$$

using expressions for $x'^i = f(x^i)$ and $x'_i = f'(x^i)$

$$= (x_1 - x_2 \cot\alpha)\, x_1 + x_2 \csc\alpha\, (x_1 \cos\alpha + x_2 \sin\alpha) =$$

$$= x_1^2 + x_2^2 \equiv |\bar{r}|^2$$

Warning:

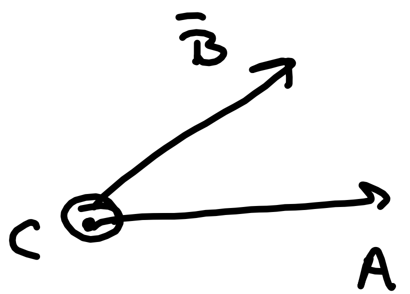
$x^i x^i \neq r^2$ $\quad x^i x^i = x_1^2 + \cot^2\alpha\, x_2^2 - 2 x_1 x_2 \cot\alpha + x^2 \csc^2\alpha \neq r^2$

$x_i x_i \neq r^2$

Vector or cross product:

$\bar{C} = \bar{A} \times \bar{B}$ $C = AB\sin\theta$

points out due to right hand rule.



In 3D:

$$\bar{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

We see that $\bar{B} \times \bar{A} = - \bar{A} \times \bar{B}$

Preview: tensor notation

$$C_i = \varepsilon_{ijk} A^j B^k$$

ijk: in cyclic order

ε_{ijk}: Levi-Civita tensor

0 if i, j or k are not all different

1 if i, j, k are in cyclic order

-1 if i, j, k are not in cyclic order.

$\varepsilon_{112} = 0 \qquad \varepsilon_{231} = 1 \qquad \varepsilon_{213} = -1$

$$C_i = \varepsilon_{ijk} A^j B^k$$

then

$$C_x \equiv C_1 = \underbrace{\varepsilon_{123}}_{1} A^2 B^3 + \underbrace{\varepsilon_{132}}_{-1} A^3 B^2 = A^2 B^3 - A^3 B^2 =$$

$$= A^y B^z - A^z B^y$$

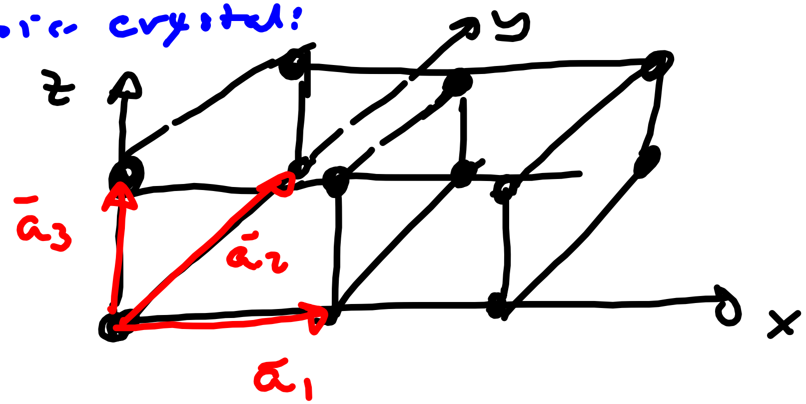
Vectors, covectors, reciprocal space and dual space.

Solid State Physics:

Crystals are form by arrays of atoms.

The position of any ion is obtained in terms of the basis vectors:

Cubic crystal:



$$\bar{a}_1 = d\,\hat{e}_1$$

$$\bar{a}_2 = d\,\hat{e}_2$$

$$\bar{a}_3 = d\,\hat{e}_3$$

d : lattice constant

$[d]$: length

$\{\bar{a}_i\}$ are linearly independent and the position $\bar{R}$ of any ion is determined by a set of 3 integers:

$$\bar{R} = n_1 \bar{a}_1 + n_2 \bar{a}_2 + n_3 \bar{a}_3$$