11/12

Tips for next weeks:

- On Thursday bring certificate for class evaluation. You'll get bonus points for class participation.
- Last homework due is Hw #11. Due on 11/26.
- Homework #12. Assigned on 11/26. Not due. Solutions posted on 11/26.
- Last day of class December 3. Come in since you'll get the take home final.
- Final: December 5 @ 2:45 PM. I'll be in room 306 @ 4:45 PM collecting them. You can bring it to me earlier if you want (give it to me in person!).

Tips for Midterm #2:

- Bring your calculator.
- Bring your book (for tables of functions).
- Provide numerical values if asked.
- Remember: $\bar{A} \times \bar{B} \longrightarrow \varepsilon_{ijk} A^j B^k$
- $x_i = g_{ij} x^j \qquad x^i = g^{ij} x_j$
- In Minkowski's space

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$\mu = \underline{\underline{0}}, 1, 2, 3$

$x_0 = ct \qquad x_3 = z$

$x_1 = x$

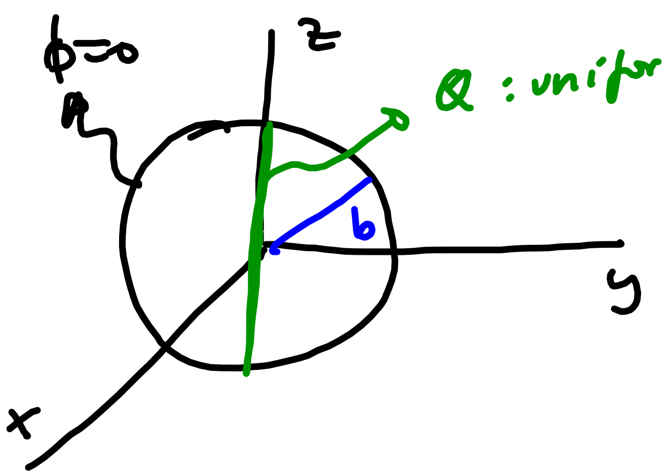
$x_2 = y$

$$A_\mu A^\mu = g_{\mu\nu} A^\nu A^\mu = (A^0)^2 - |\bar{A}|^2$$

$$\partial_\lambda = (\partial_0, \bar{\nabla}) \qquad \partial^\lambda = g^{\lambda\nu}\partial_\nu$$

- Separation of variables.
  - Do NOT solve Laplace's equation - use solutions given in class.
  - Your task is to obtain coefficients using boundary conditions.
  - Use principle of superposition if convenient.
- Be careful with signs, factors, etc.

Green Functions applications:



Find $\phi(r,\theta,\varphi)$ for $r<b$.

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V G(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,')\,d^3r'$$

$$\rho(\vec{r}\,') = \frac{Q}{2b}\,\frac{1}{2\pi r'^2}\left[\delta(\cos\theta'-1) + \delta(\cos\theta'+1)\right]$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^b r'^2 dr' \int_{-1}^{1} d(\cos\theta') \int_0^{2\pi} d\varphi' \, \rho(\vec{r}') \, G(\vec{r},\vec{r}') =$$

$$= \frac{Q}{2\pi \; 2b} \; \frac{4\pi}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} Y_{\ell m}(\theta,\varphi)$$

$$\int_0^b \left[\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}} \right] dr' \int_{-1}^{1} d(\cos\theta') \left[\delta(\cos\theta'+1) + \delta(\cos\theta'-1) \right]$$

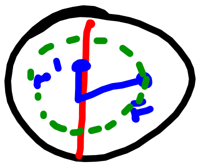
$$\underbrace{\int_0^{2\pi} d\varphi' \, Y_{\ell m}^*(\theta',\varphi')}_{2\pi \, \delta_{m,0} \sqrt{\frac{2\ell+1}{4\pi}} \, P_\ell(\cos\theta')} =$$

$$= \frac{Q}{4\pi 2 b \varepsilon_0} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \int_0^b \left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}} \right] dr'$$

$$\underbrace{\left[\overbrace{P_\ell(1)}^{1} + \overbrace{P_\ell(-1)}^{(-1)^\ell} \right]}_{\substack{0 \text{ if } \ell \text{ is odd} \\ 2 \text{ if } \ell \text{ is even}}} =$$

now $\ell = 2j$
because even ℓ terms survive

$$= \frac{Q}{4\pi b \varepsilon_0} \sum_{j=0}^{\infty} P_{2j}(\cos\theta) \underbrace{\int_0^b \left[\frac{r_<^{2j}}{r_>^{2j+1}} - \frac{r^{2j} r'^{2j}}{b^{4j+1}} \right] dr'}$$



$$\int_0^r \frac{r'^{2j}}{r^{2j+1}} dr' + \int_r^b \frac{r^{2j}}{r'^{2j+1}} dr' - \frac{r^{2j}}{b^{4j+1}} \int_0^b r'^{2j} dr'$$

$$\int_0^r \frac{r'^{2j}}{r^{2j+1}}\,dr' = \frac{1}{2j+1}\frac{r'^{2j+1}}{r^{2j+1}}\Big|_0^r = \frac{1}{2j+1}$$

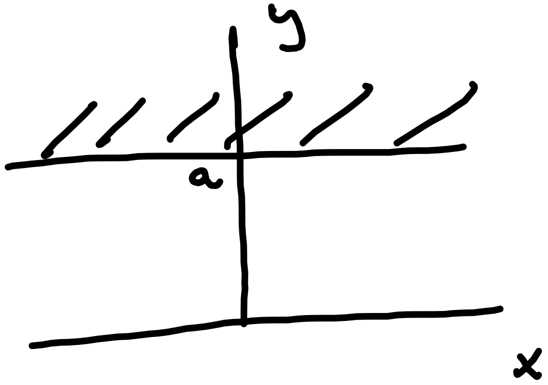
$$\int_r^b \frac{r^{2j}}{r'^{2j+1}}\,dr' = \begin{cases} \int \frac{1}{r'}\,dr' = \ln r & \text{if } j=0 \\ -\frac{r^{2j}}{2j\, r'^{2j}}\Big|_r^b = \frac{-r^{2j}}{2j\, b^{2j}} + \frac{1}{2j} & \text{if } j>0 \end{cases}$$

$$\int_0^b r'^{2j}\,dr' = \frac{1}{2j+1} r'^{2j+1}\Big|_0^b = \frac{b^{2j+1}}{2j+1}$$

Then

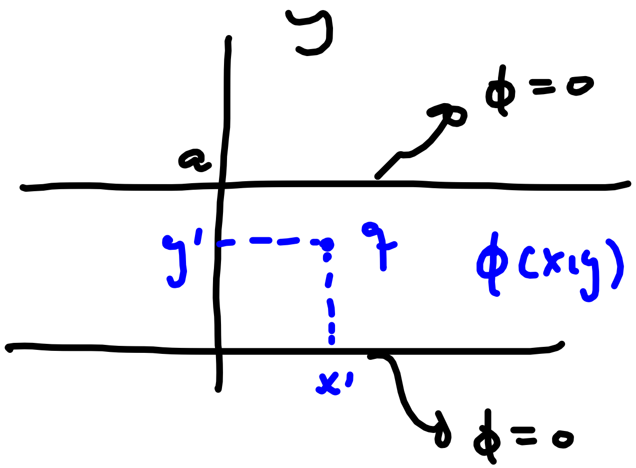
$$\phi(r,\theta) = \frac{\alpha}{4\pi b\,\varepsilon_0}\left\{\left[\sum_{j=1}^{\infty} P_{2j}(\cos\theta)\left[\frac{1}{(2j+1)} - \frac{r^{2j}}{2j\, b^{2j}} + \frac{1}{2j} - \frac{r^{2j}}{(2j+1)\, b^{2j}}\right]\right] + \ln b/r\right\}$$

- For homework:



Find $G(\bar{r}, \bar{r}')$

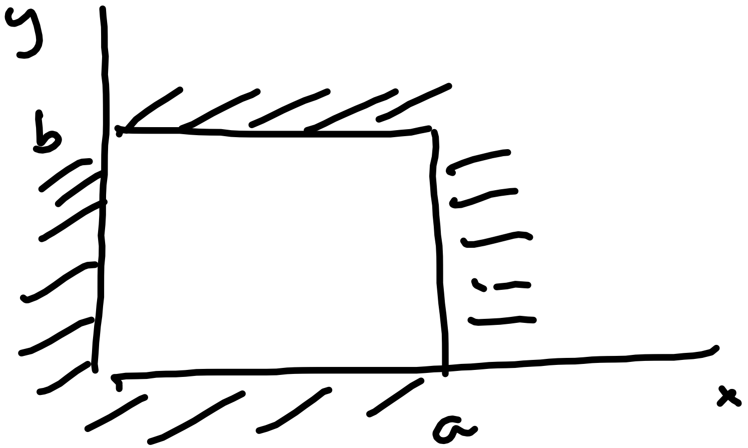
- Set $\phi = 0$ on surfaces.
- Find ϕ inside the volume for a charge $q = 4\pi\varepsilon_0$ at (x', y') using separation of variables.



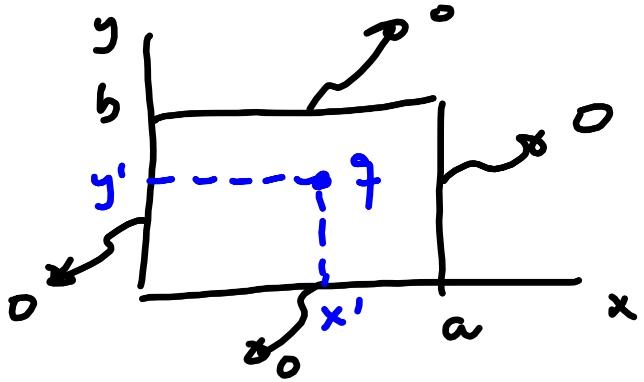
$$G(x, x', y, y') = 4 \sum_{n=1}^{\infty} \frac{e^{\frac{n\pi}{a}(x_< - x_>)}}{n} \cdot \sin\frac{n\pi y}{a} \sin\frac{n\pi y'}{a}$$

$x_<$ ($x_>$) is smaller (larger) between x and x'.

More Green functions:

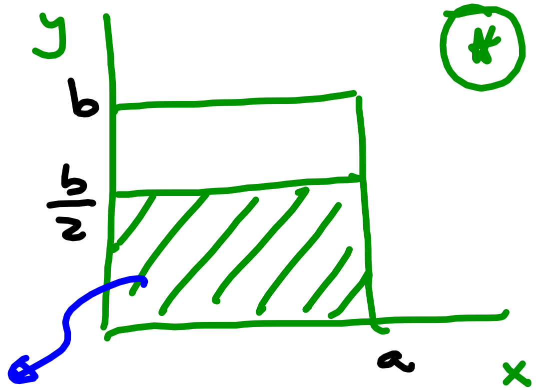


Find $G(\bar{x},\bar{x}')$:



$q = 4\pi\varepsilon_0$

We want G to solve this problem:



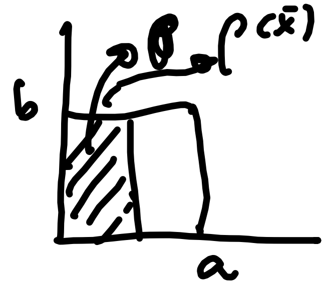
$\rho(\bar{x}) = \sigma_0\left[1-\Theta\left(y-\frac{b}{2}\right)\right]$

For this problem we will use

$$G(x, x', y, y') = 8 \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a} \frac{\sinh\left[(b-y_>)\frac{n\pi}{a}\right]}{\sinh \frac{bn\pi}{a}} \sinh \frac{n\pi y_<}{a}$$

$y_<$ ($y_>$) is the smaller (larger) between y and y'.

For this problem:



Exchange x with y and x' with y' in $G(\bar{x}, \bar{x}')$ and a with b.

To solve (*) : $\rho(x',y') = \sigma_0 \left[1 - \Theta\left(y' - \frac{b}{2}\right)\right]$

$$\Phi(x,y) = \frac{1}{4\pi\varepsilon_0} \int_0^a dx' \int_0^b dy' \, G(x,x',y,y') \, \rho(x',y') =$$

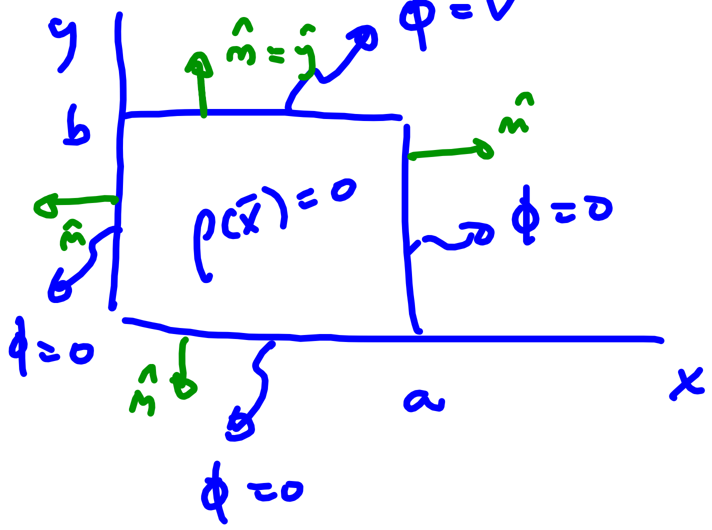
$$= \frac{1}{4\pi\varepsilon_0} \int_0^a dx' \int_0^{b/2} dy' \, G(x,x',y,y') =$$

$$= \frac{1}{4\pi\varepsilon_0} \int_0^a dx' \left[\int_0^y dy' \, G + \int_y^{b/2} dy' \, G \right]$$

with $y_< = y'$
$y_> = y$

with $y_< = y$
$y_> = y'$

Another kind of problem that you can solve:



$$\frac{\partial G}{\partial y'}\bigg|_{y'=b} \qquad G \text{ with } \; y_> = y', \; y_< = y$$

$$\phi(\bar{x}) = \underbrace{\frac{1}{4\pi\varepsilon_0}\int_V G(\bar{x},\bar{x}')\,\overset{0}{\rho(\bar{x}')}\,d^3x'}_{0} - \underbrace{\frac{1}{4\pi}\oint_S ds'\,\phi_s \frac{\partial G}{\partial n'}\bigg|_S}_{\frac{1}{4\pi}\int_0^a dx'\, V \frac{\partial G}{\partial y'}\big|_{y'=b}}$$