11/19

Integral Transforms (Ch. 20)

$$g(\alpha) = \int_a^b f(t)\, \underbrace{K(\alpha, t)}_{\text{Kernel}}\, dt$$

integral transform

function

The transform provides a mapping of a function $f(t)$ into another function $g(\alpha)$.

α and t are called conjugated variables.

Examples: x and k, ω and t.

Fourier Transforms

Kernel: $K(\alpha, t) = e^{-i\alpha t}$ or $\alpha \equiv \omega$ $K(\omega, t) = e^{-i\omega t}$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \qquad (1)$$

$g(\omega)$ provides the spectral decomposition in plane waves of $f(t)$.

$\{e^{-i\omega t}\}$ form a complete and orthogonal set of functions in $(-\infty, \infty)$:

Then:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}\,dt = \delta(\omega-\omega') \quad \text{orthogonality}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}\,d\omega = \delta(t-t') \quad \text{completitude.}$$

Antitransform:

Multiply ① by $\frac{e^{-i\omega t'}}{\sqrt{2\pi}}$ and integrate

between $(-\infty, \infty)$:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t'}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t'}\, d\omega \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, dt \underbrace{\int_{-\infty}^{\infty} e^{-i\omega(t'-t)}\, dt}_{2\pi\,\delta(t'-t)} =$$

$$= \frac{\cancel{2\pi}}{\cancel{2\pi}} f(t')$$

Then renaming $t' \to t$:

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t}\, d\omega$$

Generalization to 3D:

$$g(\bar{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\bar{r})\, e^{-i\bar{k}\cdot\bar{r}}\, d^3r$$

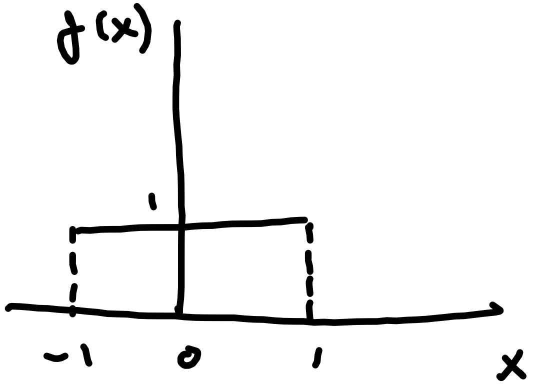
$\bar{k}\cdot\bar{r} = k_x x + k_y y + k_z z$ if $\bar{k} = (0, 0, k)$

then $\bar{k}\cdot\bar{r} = kz = kr\cos\theta$ useful to do the integral in spherical coordinates

$$f(\bar{r}) = \frac{1}{(2\pi)^{3/2}} \int g(\bar{k})\, e^{-i\bar{k}\cdot\bar{r}}\, d^3k$$

Example:

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$



$$g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{+i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{+i\alpha x}}{i\alpha} \right|_{-1}^{1} =$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i\alpha} - e^{-i\alpha}}{i\alpha} \right) = \frac{1}{\sqrt{2\pi}} \frac{2 \sin \alpha}{\alpha} = \boxed{\sqrt{\frac{2}{\pi}} \frac{\sin \alpha}{\alpha}}$$

Let's antitransform $g(\alpha)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\alpha)\, e^{-i\alpha x}\, d\alpha =$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha}\, e^{-i\alpha x}\, d\alpha =$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} (\cos\alpha x - i \sin\alpha x)\, d\alpha =$$

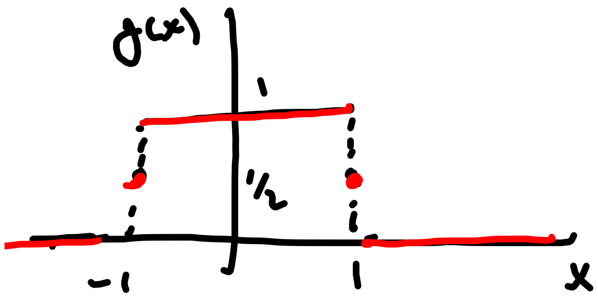
$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin\alpha \cos\alpha x}{\alpha}\, d\alpha$$

If $x = \pm 1$ then

$$f(\pm 1) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha \cos\alpha}{\alpha}\, d\alpha = \frac{1}{2}$$

average value of $f(x)$ at the discontinuity.

From $g(x)$ we obtain:



Also using this we find that

$$\int_0^\infty \frac{\sin\alpha \cos\alpha x}{\alpha}\, d\alpha = \begin{cases} \frac{\pi}{2} & \text{if } |x|<1 \\ \frac{\pi}{4} & \text{if } |x|=1 \\ 0 & \text{if } |x|>1 \end{cases}$$

$$\parallel$$

$$\frac{\pi}{2} f(x)$$

Cosine and Sine FT:

Since $e^{-ikx} = \cos kx - i \sin kx$

if we expand a function with even or odd symmetry only one of the 2 terms survives.

Cosine transform: (for even functions):

if $f_c(x) = f_c(-x)$

$$g_c(k) = \frac{1}{\sqrt{2\pi}} 2\int_0^\infty f_c(x)\cos kx \, dx = \boxed{\sqrt{\frac{2}{\pi}}\int_0^\infty f_c(x)\cos kx \, dx}$$

and

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(k) \cos kx \, dk .$$

Sine transform : (odd functions)

$$f_s(x) = -f_s(-x)$$

$$g_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(x) \sin kx \, dx$$

and

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(k) \sin kx \, dk$$

Application to problems:

1) transform the problem.

2) Solve it

3) antitransform to obtain the answer.

Fourier transform of $f'(x)$:

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$g_1(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)\, e^{-ikx}\, dx$$

If $\lim_{|x|\to\infty} f(x) = 0$ then I can integrate by parts:

$$\int_a^b u'v\,dx = uv\Big|_a^b - \int_a^b uv'\,dx$$

$$g_1(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\,e^{-ikx}\,dx = \underbrace{\frac{1}{\sqrt{2\pi}} f(x)\,e^{-ikx}\Big|_{-\infty}^{\infty}}_{0}$$

$$-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\frac{d}{dx}e^{-ikx}\,dx = -\frac{-ik}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx$$

$$= -ik\,g(k)$$

In general

$$g_n(k) = (-ik)^n \, g(k)$$

where $g_n(k) = FT\left(\dfrac{d^n f(x)}{dx^n}\right)$.

A derivative of order n in x-space becomes just a multiplication in k-space.

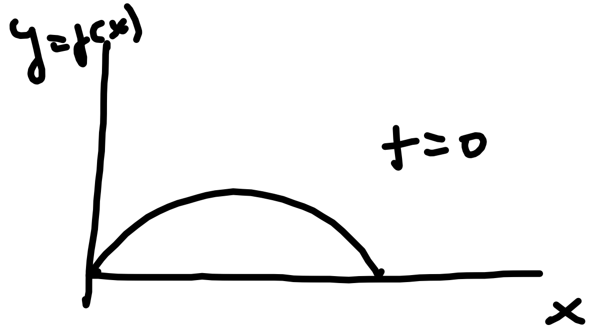
Application:

Solve differential equations:

① $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$

$y = y(x,t) = ?$

$\left.\begin{array}{l} y(t=0) \\ y'(t=0) \end{array}\right\}$ initial conditions.



$y(x,0) = f(x)$

Let's work in k- space!

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(k,t)\, e^{-ikx}\, dk \qquad ②$$

Plug ② in ①:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -k^2\, y(k,t)\, e^{-ikx}\, dk = \frac{1}{v^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 y(k,t)}{\partial t^2} e^{-ikx}\, dk$$

We see that for each k we have that

$$-k^2 y(k,t) = \frac{1}{v^2} \frac{\partial^2 y(k,t)}{\partial t^2}$$

we see that

$$y(k,t) = y_k\, e^{\pm i v k t}$$

We found that

$$y(k,t) = y_k \, e^{\pm i v k t}$$

independent of t

Also we know that for $t=0$ $y(x,0) = f(x)$

then $y_k = FT(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} dx$

and $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y_k e^{-ikx} dk$

Then:

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{y_k e^{\pm i v k t}}_{y(k,t)} e^{-ikx} dk =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y_k e^{-ik(x \pm vt)} dk = f(x \pm vt)$$

The actual solution will be

$$y(x,t) = A\, f(x+vt) + B\, f(x-vt)$$

with A and B determined by

$y(x,0)$ and $y'(x,0)$ [initial conditions].