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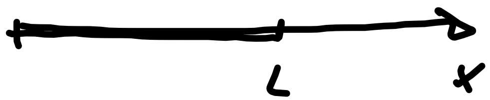
Heat flow equation

$$\frac{\partial \psi(x,t)}{\partial t} = a^2 \frac{\partial^2 \psi(x,t)}{\partial x^2} \qquad (1)$$



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Change in heat stored

$$-\frac{dQ}{dt} = -c\rho\frac{dT}{dt}$$

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Net heat out

$\bar{\nabla} Q =$

$\bar{\nabla}(-\sigma \nabla T) =$

σ const

$= -\sigma \nabla^2 T$

$\psi = T$

Let's write $\psi(x,t)$ as a FT:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(k,t)\, e^{-ikx}\, dk \qquad (2)$$

Plugging (2) in (1):

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial \psi}{\partial t}(k,t)\, e^{-ikx}\, dk = \frac{a^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -k^2 \psi(k,t)\, e^{-ikx}\, dk$$

For each k we have that:

$$\frac{\partial \psi}{\partial t}(k,t) = -a^2 k^2 \psi(k,t)$$

$$\int_{\psi(t=0)}^{\psi(t)} \frac{d\psi}{\psi} = -a^2k^2 \int_0^t dt \Rightarrow \ln \psi(k,t) - \ln \psi(k,0) = -k^2a^2t \qquad (3)$$

Exponentiating ③ we obtain:

$$\psi(k,t) = \psi(k,0)\, e^{-k^2 a^2 t} \qquad (4)$$

Plugging ④ in ②:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(k,0)\, e^{-k^2a^2t}\, e^{-ikx}\, dk$$

Now the solution depends on initial conditions.

If $\psi(k,0) = C$ then

From FT tables or using complex variable integration

$$\psi(x,t) = \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{e^{-k^2a^2t}}_{\text{gaussian}}\, e^{-ikx}\, dk = \frac{C}{a\sqrt{2t}}\, e^{-\frac{x^2}{4a^2t}}$$

Green functions for inhomogeneous Helmholtz equation.

(Ch. 10)

$$\nabla^2 \psi(\bar{x}) + k^2 \psi(\bar{x}) = -4\pi f(x) \quad (1)$$

$k=0 \rightarrow$ Poisson

Source of perturbation.

To solve (1) we will find $G(\bar{x}, \bar{x}')$ such that

$$\nabla^2 G(\bar{x}, \bar{x}') + k^2 G(\bar{x}, \bar{x}') = -4\pi \delta(\bar{x} - \bar{x}') \quad (2)$$

It can be shown that

$$\psi(\bar{x}) = \frac{1}{4\pi} \int_{\text{all space}} f(\bar{x}') G(\bar{x}, \bar{x}') d^3x'$$

for $G(\bar{x}, \bar{x}')$ satisfying: $\nabla^2 G(\bar{x}, \bar{x}') + \lambda\, q(\bar{x})\, G(\bar{x}, \bar{x}') = -4\pi \delta(\bar{x} - \bar{x}')$

We will solve ② to find $G(\bar{x}, \bar{x}')$:

Since V = all space which is isotropic then

G cannot depend on θ, θ', φ or φ' then

$$G = G(r, r') = G(|r - r'|) = G(R) \quad ③$$

Then eq. ② becomes:

$$\frac{1}{R}\frac{d^2(RG)}{dR^2} + k^2 G(R) = -4\pi\,\delta(R) \quad ⑤$$

For the homogeneous equation

$$G(R) = \frac{A e^{ikR} + B e^{-ikR}}{R}$$ is solution. ⑥

If $k \to 0$ (5) becomes Poisson's eq.

Then (6) becomes $G(R) = \frac{1}{R}$ then

$A + B = 1$

Let's propose:

$$G_k^{\pm}(R) = \frac{e^{\pm i k R}}{R}$$

$+$ for $A=1,\ B=0$

$-$ for $A=0,\ B=1$

Then

$$\boxed{\psi(r) = \frac{1}{4\pi} \int_{\text{all space}} dr' \, \frac{e^{\pm i k |r-r'|}}{|r-r'|} f(r')}$$

Inhomogeneous wave equation:

$$\nabla^2\psi(\bar{x},t) - \frac{1}{c^2}\frac{\partial^2\psi(\bar{x},t)}{\partial t^2} = -4\pi f(\bar{x},t) \quad (1)$$

To find $\psi(\bar{x},t)$ we will obtain $G(\bar{x},\bar{x}',t,t')$

and then:

$$\psi(\bar{x},t) = \frac{1}{4\pi}\int d^3x' \int dt'\, G(\bar{x},\bar{x}',t,t')\, f(\bar{x}',t')$$

all space (no surface terms).

G solves the equation:

$$\nabla^2 G(\bar{x},\bar{x}',t,t') - \frac{1}{c^2}\frac{\partial^2 G(\bar{x},\bar{x}',t,t')}{\partial t^2} = -4\pi\,\delta(\bar{x}-\bar{x}')\,\delta(t-t') \quad (2)$$

To simplify the t derivative we'll work in ω space, then:

$$\psi(\bar{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(\bar{x},\omega)\, e^{-i\omega t}\, d\omega \qquad ③$$

$$f(\bar{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\bar{x},\omega)\, e^{-i\omega t}\, d\omega \qquad ④$$

Plugs ③ and ④ in ①

$$\nabla^2\psi(\bar{x},\omega) + \underbrace{\frac{\omega^2}{c^2}}_{k^2}\,\psi(\bar{x},\omega) = -4\pi f(\bar{x},\omega) \qquad ⑤$$

Helmolz equation

Since we know G for Helmholz I can write:

$$\bar{\psi}(\bar{x},\omega) = \frac{1}{4\pi}\int d^3x' \underbrace{G(\bar{x},\bar{x}',\omega)}_{G_{k=\pm\frac{\omega}{c}}(\bar{x},\bar{x}')} f(\bar{x}',\omega)$$

We found that

$$G(\bar{x},\bar{x}',\omega) = G_k(\bar{x},\bar{x}') = \frac{e^{\pm ik|\bar{x}-\bar{x}'|}}{|\bar{x}-\bar{x}'|} \qquad k = \pm\frac{\omega}{c}$$

Now I want to find

$$G(\bar{x},\bar{x}',t,t')$$

To find $G(\bar{x},\bar{x}',t,t')$ Let's FT eq. (2) for the variable t (to ω):

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\nabla^2 G(\bar{x},\bar{x}',\omega,t')\, e^{-i\omega t}\, d\omega \; -$$

this affects only this

$$-\frac{1}{c^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial t^2}\left[G(\bar{x},\bar{x}',\omega,t')\, e^{-i\omega t}\right] d\omega =$$

$$= -4\pi\,\delta(\bar{x}-\bar{x}')\,\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\, e^{i\omega t'}\, d\omega}_{\delta(t-t')}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\nabla^2 G(\bar{x},\bar{y}',\omega,t') + \frac{\omega^2}{c^2}G(\bar{x},\bar{y}',\omega,t')\right]e^{-i\omega t}\,d\omega =$$

$$= -4\pi\,\delta(\bar{x}-\bar{x}')\,\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega t'}e^{-i\omega t}\,d\omega \qquad (6)$$

Now I can write that

$$G(\bar{x},\bar{x}',\omega,t') = \underbrace{G(\bar{x},\bar{x}',\omega)}_{\text{Helmholz Green function}}\,g(t') \qquad (7)$$

Plugging (7) in (6) we get an equation for each value of ω:

$$\nabla^2 G(\bar{x},\bar{x}',\omega)\, g(t') + \frac{\omega^2}{c^2} G(\bar{x},\bar{x}',\omega)\, g(t') =$$

$$= -4\pi\, \delta(\bar{x}-\bar{x}')\, \frac{1}{\sqrt{2\pi}}\, e^{-i\omega t'}$$

Since $\nabla^2 G(\bar{x},\bar{x}',\omega) + \frac{\omega^2}{c^2} G(\bar{x},\bar{x}',\omega) = -4\pi\, \delta(\bar{x}-\bar{x}')$

then

$$\boxed{g(t') = \frac{1}{\sqrt{2\pi}} e^{-i\omega t'}} \quad (8)$$

Then

$$G(\bar{x},\bar{x}',\omega,t') = G(\bar{x},\bar{x}',\omega)\, \frac{e^{-i\omega t'}}{\sqrt{2\pi}} = \frac{e^{\pm ik|\bar{x}-\bar{x}'|}}{|\bar{x}-\bar{x}'|}\, \frac{e^{-i\omega t'}}{\sqrt{2\pi}} \quad (9)$$

Now to obtain $G(\bar{x}, \bar{y}', t, t')$ we need to AFT

$G(\bar{x}, \bar{x}', \omega, t')$:

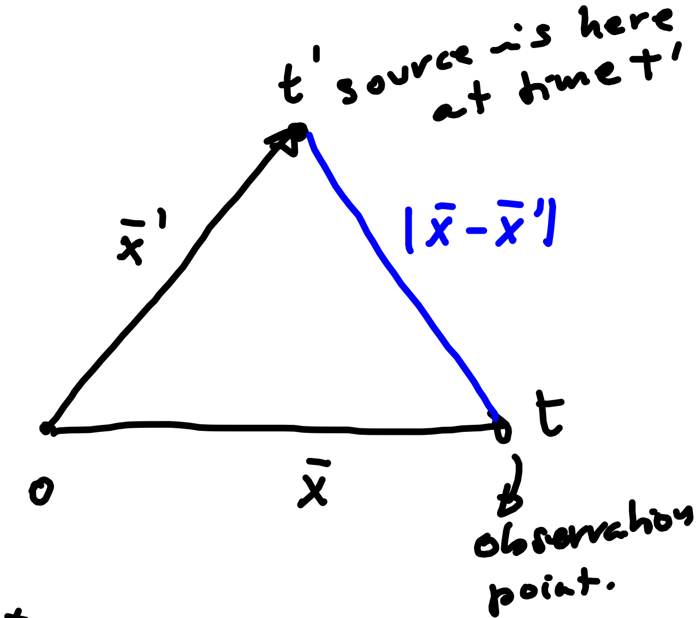
$$G^{\pm}(\bar{x}, \bar{x}', t, t') = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{\pm i \frac{\omega}{c} |\bar{x}-\bar{x}'|}}{|\bar{x}-\bar{x}'|} e^{-i\omega t'} e^{-i\omega t} \, d\omega =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega\left[t' - t \pm \frac{|\bar{x}-\bar{x}'|}{c}\right]}}{|\bar{x}-\bar{x}'|} \, d\omega = \frac{\cancel{2\pi}}{\cancel{2\pi}} \frac{\delta\left[t' - t \pm \frac{|\bar{x}-\bar{x}'|}{c}\right]}{|x - x'|}$$

We know that $\int_{-\infty}^{\infty} e^{-i\omega(t-t')} \, d\omega = 2\pi\,\delta(t-t')$

Then

$$G^{\pm}(\bar{x}, \bar{x}', t, t') = \frac{\delta\left[t' - \left[t \mp \frac{|\bar{x} - \bar{x}'|}{c}\right]\right]}{|\bar{x} - \bar{x}'|}$$



It takes a time $t - t'$ for the effect of the source at $\bar{x}'$ to reach $\bar{x}$.

if $t' < t$

$$t' = t - \frac{|\bar{x} - \bar{x}'|}{c}$$

$$t = t' + \frac{|\bar{x} - \bar{x}'|}{c}$$

G^{+} corresponds to this $(t' < t)$ and is called retarded Green function.

$G^{-}(\bar{x},\bar{x}',t,t')$ is called the advanced Greens function for $t'>t$.

How are $G^{\pm}$ used?

If at $t\to-\infty$ there is a wave $\psi_{in}(\bar{x},t)$

that satisfies $\nabla^2\psi_{in}-\frac{1}{c^2}\frac{\partial^2\psi_{in}}{\partial t^2}=0$

$\psi_{in}\propto e^{-i(\bar{k}\cdot\bar{x}-\omega t)}$ (plane wave)

Now at time t_0 we turn on a source $f(\bar{x},t)$

then:

$$\psi(\bar{x},t) = \psi_{in}(x,t) + \frac{1}{4\pi}\iint \underbrace{G^+(\bar{x},\bar{x}',t,t')}_{0 \text{ for } t<t'} j(\bar{x}',t')\, d^3x'\, dt'$$