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Note: Second Midterm does not include what we are learning today and next week.

Included:

- tensors
  - show vector equalities
  - Relativity
  - $F^{\mu\nu}$ tensor
  - Separation of variables
  - Frobenius method.

Same format as last time:

- In class part (notes + book allowed but not internet).
- At home (everything allowed except talking to other human beings).

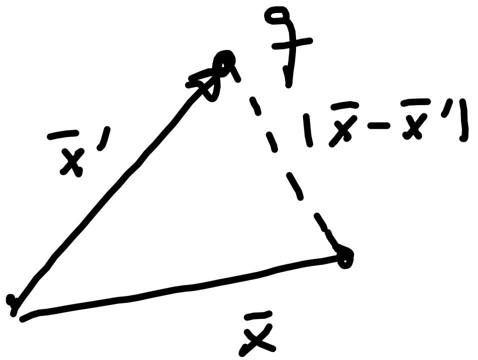
Inhomogeneous Differential Equations.

- Green's functions (Ch. 10 in the book)
  Also look at Physics Today
  Dec. 2003 page 41.

- Poisson's Equation:

$$\nabla^2 \phi(\bar{x}) = -\frac{\rho(\bar{x})}{\varepsilon_0} \quad (1)$$

For a point like charge q at $\bar{x} = \bar{x}'$ we know that (1) $\rho(\bar{x}) = q\,\delta(\bar{x} - \bar{x}')$ and $\phi(\bar{x}) = \frac{q}{4\pi\varepsilon_0|\bar{x} - \bar{x}'|}$



$$\phi(\bar{x}) = \frac{q}{4\pi\varepsilon_0 |\bar{x}-\bar{x}'|}$$

solution of Poisson's eq. for $\rho(\bar{x}) = q\,\delta(\bar{x}-\bar{x}')$.

if $q = 4\pi\varepsilon_0$ then $\phi(\bar{x}) = \frac{1}{|\bar{x}-\bar{x}'|}$ ② and

$\rho(\bar{x}) = 4\pi\varepsilon_0\,\delta(\bar{x}-\bar{x}')$. ③

Putting ② and ③ in ① we find that

$$\boxed{\nabla^2\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) = -\frac{4\pi\varepsilon_0\,\delta(\bar{x}-\bar{x}')}{\varepsilon_0} = -4\pi\,\delta(\bar{x}-\bar{x}')}$$

From elementary physics we know that if V = all space then

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \int_{V = \text{all space}} \frac{\rho(\bar{x}')\, d^3x'}{|\bar{x} - \bar{x}'|}$$

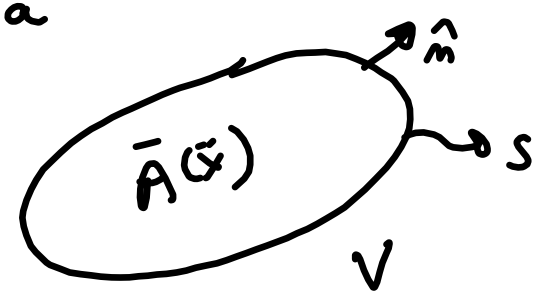
What happens if V is a finite volume?

Mathematical Detour: Green's theorem.

Consider a vector field $\bar{A}(\bar{x})$. We know that

Divergence theorem

(1) $$\int_V \bar{\nabla} \cdot \bar{A} \, d^3x' = \oint_S \bar{A} \cdot \hat{n}' da'$$



Green assumed that

(2) $\bar{A}(x) = \phi(\bar{x}) \bar{\nabla} \psi(\bar{x})$ with $\phi(\bar{x})$ and $\psi(\bar{x})$ scalar fields

Put ② in ①:

$$\bar{\nabla}\bar{A} = \bar{\nabla}(\phi\bar{\nabla}\psi) = \bar{\nabla}\phi\,\bar{\nabla}\psi + \phi\,\nabla^2\psi$$

$$\bar{A}\cdot\hat{n} = \phi(\bar{x})\,\bar{\nabla}\psi(x)\cdot\hat{n} = \phi(\bar{x})\frac{\partial\psi}{\partial n}$$

$$\int_V(\phi\,\nabla^2\psi + \bar{\nabla}\phi\,\bar{\nabla}\psi)\,d^3x' = \oint_S \phi\frac{\partial\psi}{\partial n'}\,da' \quad ③$$

Then Green proposed that

$\bar{A}(\bar{x}) = \psi(\bar{x})\,\bar{\nabla}\phi(\bar{x})$ replace this in ①:

$$\int_V(\psi\,\nabla^2\phi + \bar{\nabla}\psi\,\bar{\nabla}\phi)\,d^3x' = \oint_S \psi\frac{\partial\phi}{\partial n'}\,da' \quad ④$$

Substract ④ from ③:

$$\boxed{\int_V \left[\phi\, \nabla^2 \psi - \psi\, \nabla^2 \phi\right] d^3x' = \oint_S \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'}\right) da'}$$

⑤

Green's theorem.

Now let's go back to solving Poisson's eq.

Assume that $\psi = \dfrac{1}{|\bar{x} - \bar{x}'|}$ and $\phi(\bar{x})$ is the electrostatic potential and replace in ⑤:

$$\int_V \Big[\phi(\bar{x}') \underbrace{\nabla_{x'}^2 \frac{1}{|\bar{x}-\bar{x}'|}}_{-4\pi\,\delta(\bar{x}-\bar{x}')} + \frac{1}{|\bar{x}-\bar{x}'|} \underbrace{\nabla_{x'}^2 \phi}_{-\rho(\bar{x}')/\varepsilon_0} \Big] d^3x' =$$

$$= \oint_S \Big[\phi(\bar{x}') \frac{\partial}{\partial m'} \left(\frac{1}{|\bar{x}-\bar{x}'|} \right) - \frac{1}{|\bar{x}-\bar{x}'|} \frac{\partial \phi}{\partial m'} \Big] da'$$

$$\int_V \Big[-4\pi\, \phi(\bar{x}')\, \delta(\bar{x}-\bar{x}') - \frac{\rho(\bar{x}')}{\varepsilon_0 |\bar{x}-\bar{x}'|} \Big] d^3x' =$$

$$= \oint_S \Big[\phi(\bar{x}') \frac{\partial}{\partial m'} \left(\frac{1}{|\bar{x}-\bar{x}'|} \right) - \frac{1}{|\bar{x}-\bar{x}'|} \frac{\partial \phi}{\partial m'} \Big] da'$$

if $x' \in V$ then $\int_V -4\pi\, \phi(\bar{x}')\, \delta(\bar{x}-x')\, d^3x' = -4\pi\, \phi(\bar{x})$
or 0 if $x' \notin V$.

Then I see that:

$$\phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\bar{x}')}{|\bar{x}-x'|}\,d^3x' + \frac{1}{4\pi}\oint_S\left[\frac{1}{|\bar{x}-\bar{x}'|}\frac{\partial\phi}{\partial n'} - \phi\,\frac{\partial}{\partial n'}\left[\frac{1}{|\bar{x}-\bar{x}'|}\right]\right]da'$$

Valid expression but I cannot use it to find ϕ since if I know $\frac{\partial\phi}{\partial n}\Big|_S$ I don't know $\phi|_S$ or vice versa.

Notice that if $V \to$ all space $\oint_S (\ ) = 0$ and we obtain the elementary expression.

But if V is finite we need to calculate $\oint_S (\ )$.

Boundary conditions: 2 types in electrostatics:

1) Dirichlet b.c.: $\phi|_S$ provided

2) von Neumann b.c.: $\left.\frac{\partial\phi}{\partial n}\right|_S$ provided

$$\left.\frac{\partial\phi}{\partial n}\right|_S = E_n|_S$$

To satisfy the b.c. we are going to add to $\psi(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|}$ a term

$F(\bar{x},\bar{x}')$ such that $\nabla^2_{x'} F(\bar{x},\bar{x}') = 0$ inside V.

I define :

$$\psi = G(\bar{x}, \bar{x}') = \frac{1}{|\bar{x} - \bar{x}'|} + F(\bar{x}, \bar{x}')$$

notice that :

$$\nabla^2 \psi = \nabla^2 G(\bar{x}, \bar{x}') = \underbrace{\nabla^2_{x'}\left(\frac{1}{|x - x'|}\right)}_{-4\pi\,\delta(\bar{x} - \bar{x}')} + \underbrace{\nabla^2_{x'} F(\bar{x}, \bar{x}')}_{0} =$$

$$= -4\pi\,\delta(\bar{x} - \bar{x}') \quad \text{inside } V.$$

$G(\bar{x}, \bar{x}')$ is the Green function.

Plugging $\psi = G(\bar{x},\bar{x}')$ in (5) we obtain:

$$\boxed{\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\bar{x}')\, G(\bar{x},\bar{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \left(G \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial G}{\partial n'} \right) da'} \quad (6)$$

If we obtain $G(\bar{x},\bar{x}')$ for a given geometry (volume V) we can find $\Phi(\bar{x})$ in V for any $\rho(\bar{x}')$ and any b.c..

- For Dirichlet b.c. since I know $\phi|_S$ but I do not know $\frac{\partial \phi}{\partial n}\Big|_S$ we request $G|_S = 0$ so that term vanishes in (6).

- For von Neumann b.c. we know $\frac{\partial \phi}{\partial n}\Big|_S$ then we request that

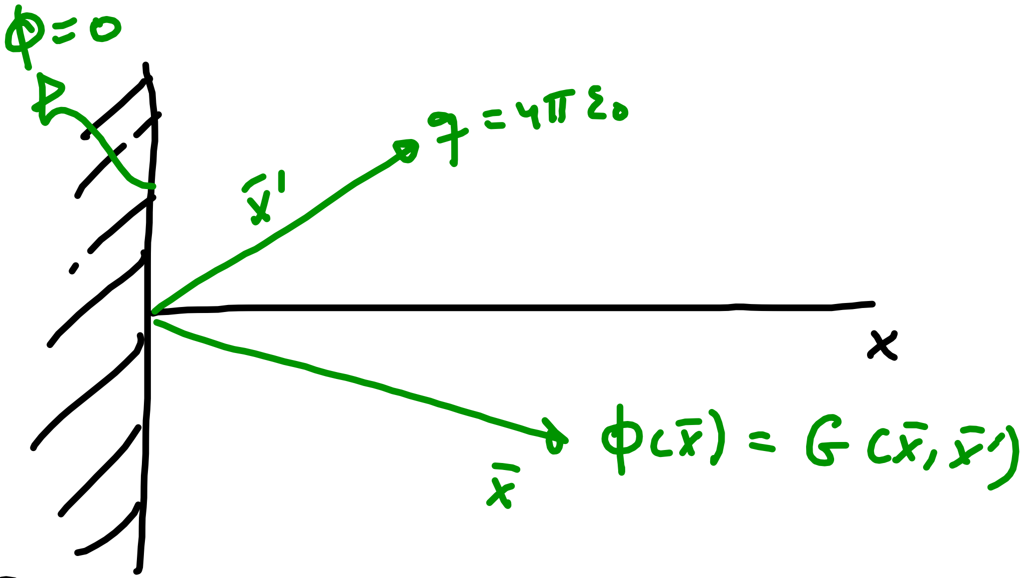
$$\frac{\partial G_N}{\partial n}\Big|_S = -\frac{4\pi}{S} \rightarrow 0 \text{ if } S \rightarrow \infty$$

$G(\bar{x}, \bar{x}')$ is the potential of a point charge $q = 4\pi\varepsilon_0$ at $\bar{x}'$ inside V plus the potential of outside charges $\longrightarrow$

that insure that the b.c.'s are satisfied.

Example:

V: (x, y, z) with $x \geq 0$.



Find $G(\bar{x}, \bar{x}')$ with Dirichlet b.c. for this geometry.

Propose:

$$G(\bar{x}, \bar{x}') = \frac{1}{|\bar{x} - \bar{x}'|} + \frac{q'}{|\bar{x} - \bar{x}''| 4\pi\varepsilon_0}$$

$\bar{x}''$ outside V with $x < 0$.

Notice that $G(\bar{x}, \bar{x}')\big|_S = 0$ then

q' has to be negative. If $q' = -4\pi\varepsilon_0$ then

$$G(\bar{x}, \bar{x}') = \frac{1}{|\bar{x} - \bar{x}'|} - \frac{1}{|\bar{x} - \bar{x}''|}$$

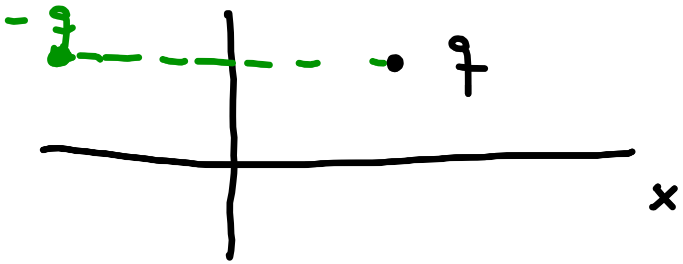
at $\bar{x} = (0, y, z)$:

$$0 = \frac{1}{\sqrt{(0-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(0-x'')^2 + (y-y'')^2 + (z-z'')^2}}$$

Then $x'' = -x'$ $\quad y'' = y'$ $\quad z'' = z'$

Then:

$$\boxed{G(\bar{x},\bar{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}}}$$



$-q$ is the image charge of q and produces the potential that satisfies the b.c.

Now you can find $\phi(\bar{x})$ for any charge distribution for $x \geq 0$.

Example 1: q at $(d,0,0)$. $V=0$ on surface.

$$\phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \underbrace{\rho(\bar{x})}_{q\,\delta(x-d)\,\delta(y)\,\delta(z)} G(\bar{x},\bar{x}')\, d^3x' + 0$$

$\phi_s(\ ) = 0$ since $V=0$ on surface and $G=0$ on surface.

$$= \frac{1}{4\pi\varepsilon_0}\int_0^\infty dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz'\, q\,\delta(x'-d)\,\delta(y')\,\delta(z')$$

$$\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}}\right] =$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right]$$

Also: If V on surface is $f(y,z)$ then

$$\Phi(\bar{x}) = \int_V (\quad) + \frac{1}{4\pi}\int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz'\, f(y',z') \left.\frac{\partial G}{\partial(-x')}\right|_S$$

if $f(y,z) = V_0$ then you can show that

$$\iint \frac{\partial G}{\partial n'}\, da = 4\pi$$

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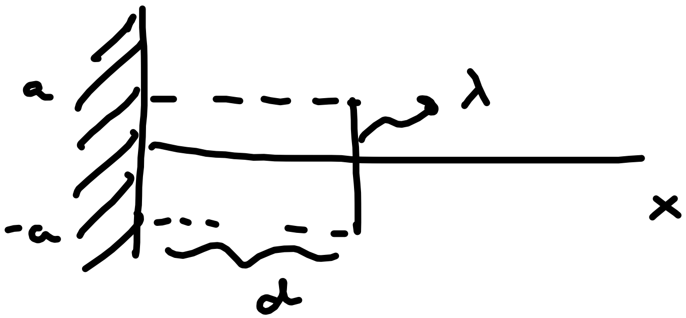
$\phi(\bar{x}) = ?$



$$\rho(\bar{x}) = \lambda\, \delta(x-d)\, \delta(z)\, u(y+a)\,[1 - u(y-a)]$$

$$u(t-k)\begin{cases} 0 & t<k \\ 1 & t>k \end{cases}$$