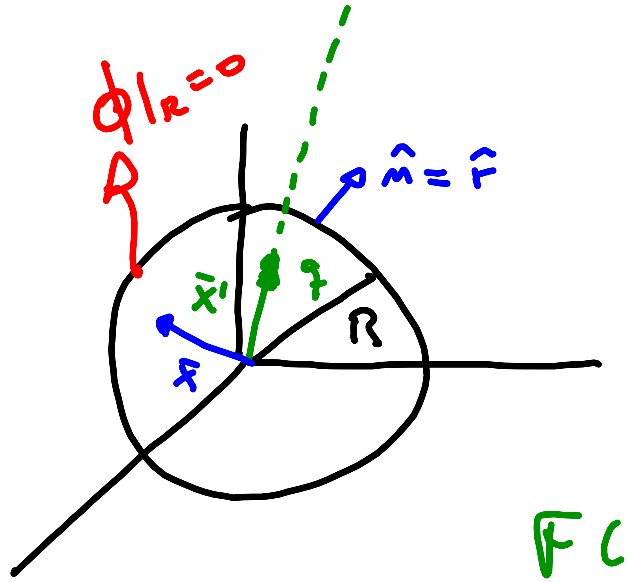
11/7

Green Functions

Spherical symmetry:



$q = 4\pi\varepsilon_0$

We want $G(\bar{x}, \bar{x}')$ for $r \leq R$ (V is volume inside the sphere).

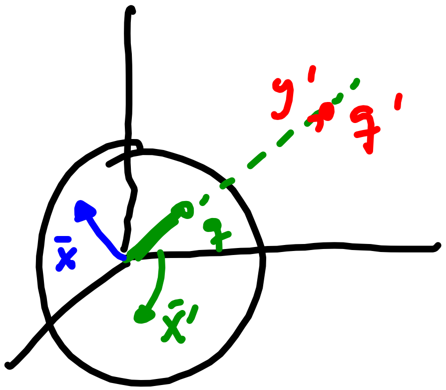
Dirichlet b.c.

$$G(\bar{x}, \bar{x}') = \frac{1}{|\bar{x} - \bar{x}'|} + F(\bar{x}, \bar{x}')$$

$F(\bar{x}, \bar{x}')$ is the potential of external charges that satisfy $G|_{x=R} = 0$.

$$G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} + \frac{q'}{4\pi\varepsilon_0 |\bar{x}-\bar{y}'|} =$$

we need to find q' and $\bar{y}'$ so that $G|_R = 0$.



$$= \frac{1}{|x\hat{m} - x'\hat{m}'|} + \frac{q'}{4\pi\varepsilon_0 |x\hat{m} - y'\hat{m}'|}$$

$$G(\bar{x},\bar{x}')\Big|_R = 0 = \frac{1}{|R\hat{m} - x'\hat{m}'|} + \frac{q'}{4\pi\varepsilon_0 |R\hat{m} - y'\hat{m}'|}$$

$$\hat{m} = \frac{\bar{x}}{x} \qquad \hat{m}' = \frac{\bar{x}'}{x'}$$

$$0 = \frac{1}{R\left|\hat{m} - \frac{x'}{R}\hat{m}'\right|} + \frac{q'}{4\pi\varepsilon_0 y' \left|\frac{R}{y'}\hat{m} - \hat{m}'\right|}$$

We request that

$$\frac{1}{R} + \frac{q'}{4\pi\varepsilon_0 y'} = 0 \quad \therefore \quad \boxed{\frac{1}{R} = -\frac{q'}{4\pi\varepsilon_0 y'}} \quad (1)$$

and

$$\left|\hat{m} - \frac{x'}{R}\hat{m}'\right| = \left|\frac{R}{y'}\hat{m} - \hat{m}'\right|$$

$$\left(\hat{m} - \frac{x'}{R}\hat{m}'\right)\cdot\left(\hat{m} - \frac{x'}{R}\hat{m}'\right) = \left(\frac{R}{y'}\hat{m} - \hat{m}'\right)\cdot\left(\frac{R}{y'}\hat{m} - \hat{m}'\right)$$

$$\cancel{\underbrace{\hat{m}\cdot\hat{m}}_{1}} - 2\hat{m}\cdot\hat{m}'\frac{x'}{R} + \frac{x'^2}{R^2}\underbrace{\hat{m}'\cdot\hat{m}'}_{1} = \frac{R^2}{y'^2}\underbrace{\hat{m}\cdot\hat{m}}_{1} - 2\hat{m}'\cdot\hat{m}\frac{R}{y'} + \cancel{\underbrace{\hat{m}'\cdot\hat{m}'}_{1}}$$

$$\boxed{\frac{x'}{R} = \frac{R}{y'}} \Rightarrow \boxed{y' = \frac{R^2}{x'}} \quad (2)$$

Plugging ② in ① we find q':

$$\boxed{q' = -\frac{4\pi\varepsilon_0 R}{x'}} \quad (3)$$

Then replacing y' and q' in G:

$$\boxed{G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} - \frac{R}{x'\left|\bar{x} - \frac{R^2}{x'}\hat{n}'\right|}}$$

- $\bar{x} = (r,\theta,\varphi)$ best option for coordinates
- $\left.\frac{\partial G}{\partial n'}\right|_R \equiv \left.\frac{\partial G}{\partial r'}\right|_R$
- Expand in terms of $Y_\ell^m(\theta,\varphi)$.

We found that

$$\frac{1}{|\bar{x}-\bar{x}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_{\ell m}^*(\theta',\varphi') Y_{\ell m}(\theta,\varphi)$$

$r_<$ ($r_>$) smaller (larger) between x and x',

$$\frac{R}{x'|\bar{x} - \frac{R^2}{x'}\hat{n}'|} = \frac{R}{x'} 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_{\ell m}^*(\ ) Y_{\ell m}(\ )$$

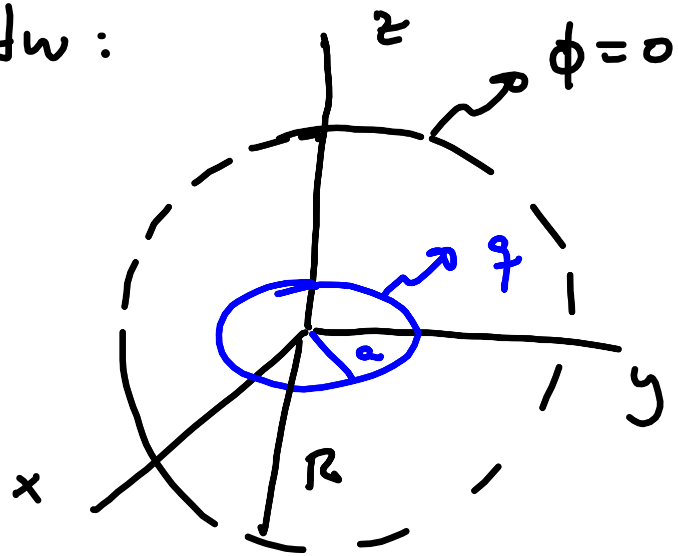
$r_<$ ($r_>$) smaller (larger) between (x and $\frac{R^2}{x'}$).

Try at home:

$$G(\bar{r},\bar{r}') = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \left(\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{r^{\ell} r'^{\ell}}{R^{2\ell+1}} \right) Y_{\ell m}(\theta,\varphi) Y_{\ell m}^*(\theta',\varphi')$$

Now with $G(\bar{r},\bar{r}')$ you can solve several problems:

Hw:



$$\rho(\bar{x}) = \frac{q}{2\pi a^2}\,\delta(r-a)\,\delta(\cos\theta)$$

Plug G and ρ in expression for $\phi(\bar{r})$ obtained last time.

$$\phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{r}')\,G(\bar{r},\bar{r}')\,d^3r'$$

$$\Phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \, 4\pi \frac{q}{2\pi a^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} Y_{\ell m}(\theta,\varphi)$$

$$\int_0^R r'^2 \, dr' \, \delta(r'-a) \left(\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{r^{\ell} r'^{\ell}}{R^{2\ell+1}} \right)$$

$$\underbrace{\int_{-1}^{1} d(\cos\theta') \, \delta(\cos\theta') \int_0^{2\pi} d\varphi' \, Y^*_{\ell m}(\theta',\varphi')}_{\delta_{m,0} \, 2\pi \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)}$$

$$\underbrace{\int_0^{2\pi} d\varphi' \, Y^*_{\ell m}(\theta',\varphi')}_{2\pi \, \delta_{m,0}}$$

since $Y^*_{\ell m} \propto e^{-im\varphi'}$

$$Y^*_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \, P_\ell^m(\cos\theta) \, e^{-im\varphi}$$

Then the integral over θ' and φ' gives us:

$$P_\ell(0)\sqrt{\frac{2\ell+1}{4\pi}}\,2\pi\,\delta_{m,0} \qquad \underbrace{\sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell(\cos\theta)}_{Y_{\ell 0}(\theta,\varphi)}$$

Then:

$$\phi(\vec r) = \frac{2\pi}{\cancel{4\pi}\,\varepsilon_0}\,\cancel{4\pi}\,\frac{q}{2\pi a^2}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\,Y_{\ell 0}(\theta,\varphi)$$

$$\underbrace{\int_0^R r'^2\,dr'\,\delta(r'-a)\left(\frac{r_<^{\ell}}{r_>^{\ell+1}}-\frac{r^\ell r'^\ell}{R^{2\ell+1}}\right)}_{\Pi}\,\overbrace{P_\ell(0)}^{0 \text{ for } \ell \text{ odd}}\sqrt{\frac{2\ell+1}{4\pi}}$$

$$P_{2j}(0) = \frac{(-1)^j(2j)!}{2^{2j}(j!)^2}$$

Let's consider II: (replacing ℓ by $2j$)

For $r > a$ $\quad r_< = r'$ and $r_> = r$.

$$II = a^{2j+2}\left(\frac{1}{r^{2j+1}} - \frac{r^{2j}}{R^{4j+1}}\right)$$

For $r < a$ $\quad r_< = r$ and $r_> = r'$:

$$II = a^{2}\left(\frac{r^{2j}}{a^{2j+1}} - \frac{r^{2j}\,a^{2j}}{R^{4j+1}}\right)$$

Then:

Then we obtain:

$$\phi^{r>a}(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{j=0}^{\infty} P_{2j}(\cos\theta) \frac{(-1)^j (2j)!}{2^{2j} (j!)^2} a^{2j} \left(\frac{1}{r^{2j+1}} - \frac{r^{2j}}{R^{4j+1}} \right)$$

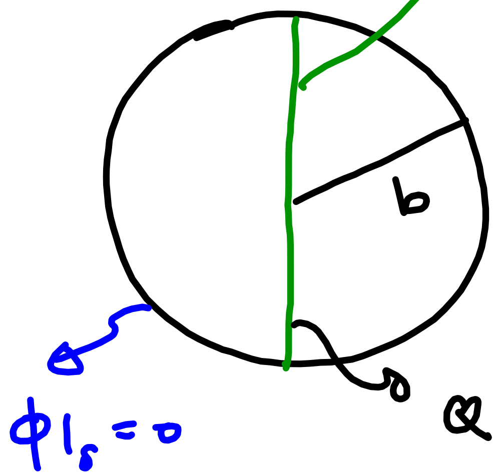
$$\phi^{r<a}(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{j=0}^{\infty} P_{2j}(\cos\theta) \frac{(-1)^j (2j)!}{2^{2j} (j!)^2} r^{2j} \left(\frac{1}{a^{2j+1}} - \frac{a^{2j}}{R^{4j+1}} \right)$$

or

$r_<$ ($r_>$) smaller (larger) between r and a.

$$\phi(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{j=0}^{\infty} P_{2j}(\cos\theta) \frac{(-1)^j (2j)!}{2^{2j} (j!)^2} r_<^{2j} \left(\frac{1}{r_>^{2j+1}} - \frac{r_>^{2j}}{R^{4j+1}} \right)$$

Next time:



$$\rho(\bar{r}') = \frac{Q}{2b}\frac{1}{2\pi r'^2}\left[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)\right]$$

$$\int_V \rho(r')\, d^3r' = Q \quad \text{(check!)}$$

Find $\phi(\bar{r})$ inside the grounded sphere.

$$\phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_V G(\bar{r}, \bar{r}')\rho(\bar{r}')\, d^3r'.$$