10/1

Dual Tensors

If C^{jk} is an antisymmetric tensor of rank 2 in 3D. It only has 3 independent components. Thus, a 3D vector can contain the same information. This vector C_i will be the Dual tensor of C^{jk}.

Let's define:

$$C_i = \frac{1}{2} \varepsilon_{ijk} C^{jk}$$

$$C^{jk} = \begin{pmatrix} 0 & C^{12} & C^{13} \\ -C^{12} & 0 & C^{23} \\ -C^{13} & -C^{23} & 0 \end{pmatrix}$$

$$C_1 = \frac{1}{2}\varepsilon_{1jk}C^{jk} = \frac{1}{2}\left(\underbrace{\varepsilon_{123}}_{1}C^{23} + \underbrace{\varepsilon_{132}}_{-1}C^{32}\right) =$$

$$= C^{23}$$

Doing the same we obtain

$$C_2 = C^{13} \quad \text{and} \quad C_3 = C^{12}$$

In 3D physics all the vectors that result from the cross-product of two vectors are actually the duals of antisymmetric tensors of rank 2.

Ex:

$$\bar{L} = \bar{r} \times \bar{p}$$

$$L_i = \varepsilon_{ijk} r^j p^k = \frac{1}{2} \varepsilon_{ijk} \underbrace{(r^j p^k - r^k p^j)}$$

$$T^{jk} = r^j p^k$$

Notice that this correspondence only works in $D=3$.

- In $D=2$ $v^i = (v^1, v^2)$ 2 elements

and $A^{ij} = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}$ 1 independent component

No match.

- In $D=4$ $v^i = (v^1, v^2, v^3, v^4)$ 4 elements

$$A^{ij} = \begin{pmatrix} 0 & a^{12} & a^{13} & a^{14} \\ & 0 & a^{23} & a^{24} \\ & & 0 & a^{34} \\ & & & 0 \end{pmatrix}$$ 6 independent components.

Another example of dual tensors:

Consider:

3 D

$$V^{ijk} = A^i B^j C^k$$

rank 3 tensor

Define:

$$V = \varepsilon_{ijk} V^{ijk} = \varepsilon_{ijk} A^i B^j C^k =$$

$$= A^i \underbrace{\varepsilon_{ijk} B^j C^k} = A^i (\bar{B} \times \bar{C})_i =$$

$$= \bar{A} \cdot (\bar{B} \times \bar{C})$$

Using tensors to derive vector calculus identities:

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{A}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2\bar{A} \quad (1)$$

$$\varepsilon^{rmi}\partial_m \underbrace{(\bar{\nabla}\times\bar{A})_i}_{\varepsilon_{ijk}\partial^j A^k} = \varepsilon^{rmi}\partial_m \varepsilon_{ijk}\partial^j A^k =$$

$$= \underbrace{\varepsilon^{irm}\varepsilon_{ijk}}_{(\delta^r_j\delta^m_k - \delta^r_k\delta^m_j)} \partial_m \partial^j A^k = \delta^r_j\delta^m_k \partial_m\partial^j A^k - \delta^r_k\delta^m_j \partial_m\partial^j A^k =$$

$$= \partial_m\partial^r A^m - \partial_m\partial^m A^r = \partial^r\partial_m A^m - \partial_m\partial^m A^r =$$

$$= [\bar{\nabla}(\bar{\nabla}\cdot\bar{A})]^r - [\nabla^2\bar{A}]^r \quad (1)$$

In the same way you can show that

$$\bar{\nabla}\cdot(\bar{a}\times\bar{b}) = \bar{b}\cdot(\bar{\nabla}\times\bar{a}) - \bar{a}\cdot(\bar{\nabla}\times\bar{b})$$

$$\underbrace{\partial_i \varepsilon^{ijk} a_j b_k}_{} = \varepsilon^{ijk}\partial_i a_j \, b_k + \varepsilon^{ijk} a_j \partial_i b_k =$$

$$\ldots$$

Tensors in Relativity (Ch. 17.8-9)

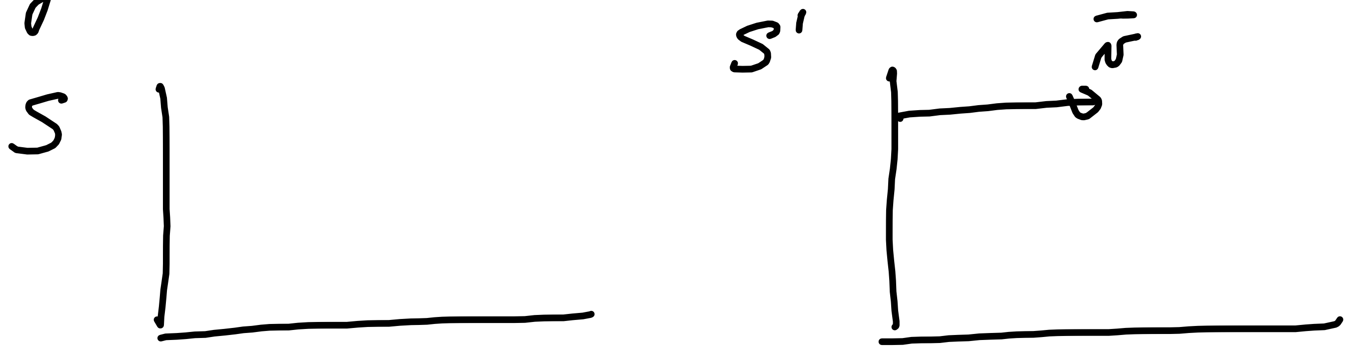
- The book uses SI units but in relativity is simpler to use Gaussian units (I'll give the conversions).

- The laws of Physics have to be the same regardless of the frame of reference.

- Invariance under space and time translations (Space and time are homogeneous).

- Invariance under rotations in 3D because space is isotropic.
- Invariance under Lorentz transformations because Maxwell's equations satisfy that.

If



So that at $t=0$ $S=S'$.

If we start a flashlight at the origin at $t=0$ it travels a distance

$r = ct$ in S and $r' = ct'$ in S'.

- c is the same (invariant) in all reference frames. Then $ct - r$ and $ct' - r'$ have to be invariants.

- Under a Lorentz transformation

$$c^2 t^2 - x^{1\,2} - x^{2\,2} - x^{3\,2} = c^2 t'^2 - x'^{1\,2} - x'^{2\,2} - x'^{3\,2}$$

Let's define $x^0 = ct$ and $x^{0'} = ct'$

Then

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

$$\parallel$$

$$dr_\mu dr^\mu = \underbrace{\bar{\varepsilon}_i \cdot \bar{\varepsilon}_j}_{g_{ij}} dx^i dx^j = g_{ij} dx^i dx^j$$

Then

$$g_{ij} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

metric tensor.

$$g^{ij} = g_{ij}$$

Since $g_{ij} g^{jk} = \delta_i^{\ k}$

Let's define

$$x^\mu = (x^0, \bar{x})$$ contravariant 4-vector.

$x^0 \to$ scalar in 3D

$\bar{x} \to$ 3D vector

then

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -\bar{x})$$ covariant 4-vector.

Scalar product:

$$x_\mu x^\mu = g_{\mu\nu} x^\nu x^\mu = (x^0)^2 - |\bar{x}|^2$$

Derivatives:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\partial_0, \bar{\nabla}\right)$$ covariant derivative

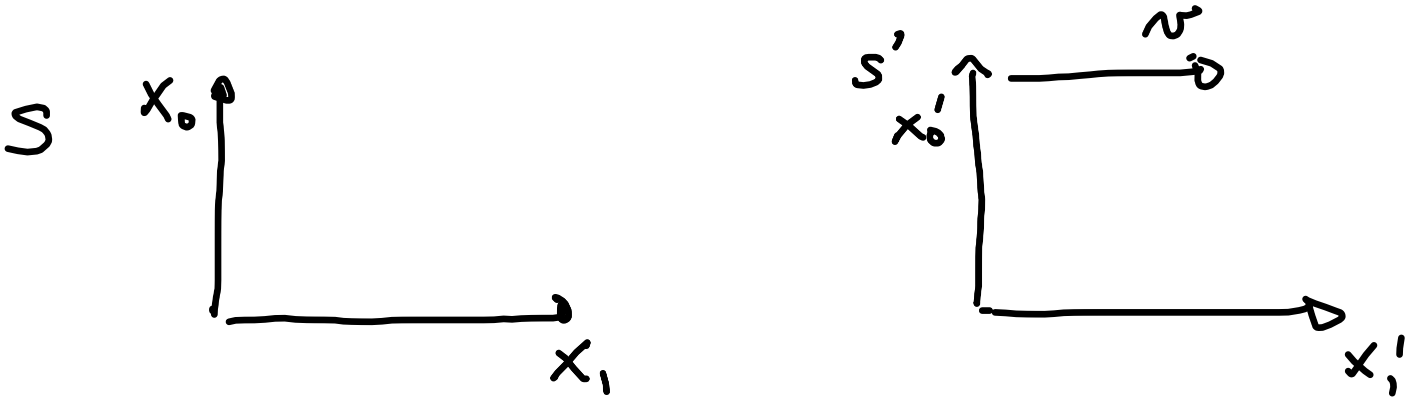
then

$$\partial^\mu = g^{\mu\nu}\partial_\nu = \left(\partial_0, -\bar{\nabla}\right)$$ contravariant derivative.

d'Alambertian (generalized Laplacian)

$$\Box = \partial^2 = \partial_\mu \partial^\mu = \left(\partial_0^2, -\nabla^2\right)$$

Lorentz Transformation:



For a spherical wave $ct=r$ and $ct'=r'$. Using this if $\bar{v} \parallel x_1$ we obtain:

$$\beta = \frac{v}{c} \leq 1 \qquad \gamma = (1-\beta^2)^{-1/2}$$

$$x'^0 = \gamma x^0 - \beta\gamma x^1$$

$$x'^1 = -\beta\gamma x^0 + \gamma x^1$$

$$x^{2'} = x^2$$

$$x^{3'} = x^3 \qquad (1)$$

From ①

element (0,0) element (1,2)

$$M^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Remember that $\mu, \nu = 0, 1, 2, 3$.

Some people defines:

$$\left.\begin{aligned} \gamma &= \cosh\rho \\ \beta\gamma &= \sinh\rho \\ \tanh\rho &= \beta = v/c \end{aligned}\right\}$$

looks like a rotation by an imaginary angle $i\rho$.

Then an alternative way of defining the variables is:

$$x^1 = x \qquad x^2 = y \qquad x^3 = z \qquad x^4 = ict$$

$$ds^2 = dx^{1\,2} + dx^{2\,2} + dx^{3\,2} + dx^{4\,2}$$

and

$$M^a{}_b = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma & i\beta\gamma \\ & & -i\beta\gamma & \gamma \end{pmatrix}$$

with

$\gamma = \cos\theta$

$i\beta\gamma = \sin\theta$

$\tan\theta = i v/c$

For test on Thursday October 3:

- Bring paper
- Bring calculator. (not your phone).
- Simplify all expressions:

$$10 \cos\frac{\pi}{3} = 10 \times \frac{1}{2} = 5$$

- Write p^{ik} in terms of p'^{jl} means:

$$p^{ik} = \frac{\partial x^i}{\partial x'^j} \cdot \frac{\partial x^k}{\partial x'^l} \, p'^{jl}$$

- Write p'^{jl} in terms of p^{ik}:

$$p'^{jl} = \frac{\partial x'^j}{\partial x^i} \frac{\partial x'^l}{\partial x^k} p^{ik}$$