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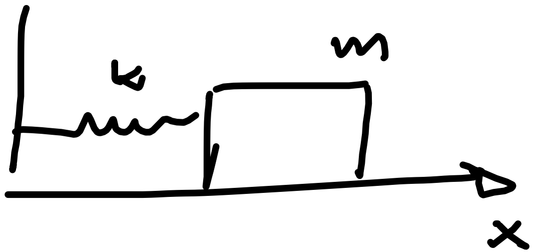
Boundary Conditions: (BC).

BC's are necessary to determine the values of the coefficients when we solve D.E.'s.

Examples:

Harmonic oscillator.



$$m\ddot{x} + kx = 0 \qquad (1)$$

$$x = A\cos(\omega t + \varphi)$$

is solution if $\omega = \sqrt{\frac{k}{m}}$

$x(0) = x_0$ and $v(0) = v_0$ initial conditions

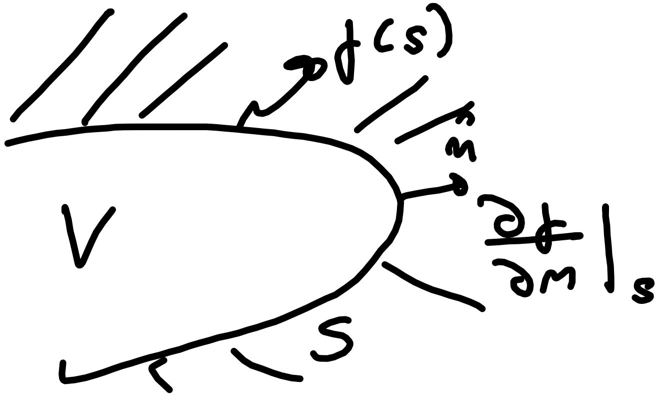
Replacing v_0 and x_0 in 1 for $t=0$

you obtain

$$\text{and}\ \begin{cases} \varphi = \tan^{-1}\left(-\dfrac{v_0}{x_0\omega}\right) \\ A = \dfrac{x_0}{\cos\varphi} \end{cases}$$

Problems defined inside finite volumes:

1) Cauchy b.c.: $f(S)$ and $\left.\frac{\partial f}{\partial n}\right|_S$ are specified at the boundary. We are trying to find $f(\bar{x})$ inside the volume defined by the surface.



2) In some cases, particularly electrostatics in close volumes if one provides both $f(s)$ and $\left.\frac{\partial f}{\partial n}\right|_s$ the solution is inconsistent then in this case there are two kinds of b.c.:

a) Dirichlet: $f(s)$ is provided.

b) von Neumann: $\left.\frac{\partial f}{\partial \hat{n}}\right|_s$ is provided.

Separation of variables.

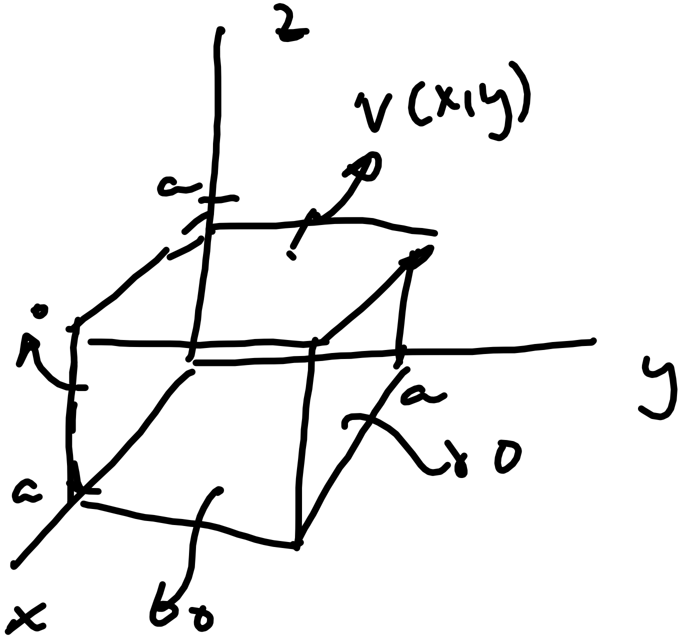
A partial differential eq. in n variables will be separated in n ordinary diff. eq.s. Each separation will introduce a separation constant determined by the b.c. For n variables there will be n-1 separation constants.

- The solution of the equation will depend on the geometry - The geometry of the surfaces that define the volume determine the coordinates that we will use.

- Laplace equation:

① $\nabla^2 \phi = 0$ homogeneous.

We want to solve ① in a volume given by $0 \leq x \leq a$ $0 \leq y \leq a$ $0 \leq z \leq a$



$$\phi(0,y,z) = \phi(a,y,z) =$$
$$= \phi(x,0,z) = \phi(x,a,z) =$$
$$= \phi(x,y,0) = 0$$

$$\phi(x,y,a) = V(x,y)$$

Warning: if $V \neq 0$ in 2 or more faces you solve the problem for each single face $\neq 0$ and add the solutions using the superposition principle.

We will assume that

$$\phi(x, y, z) = X(x)\,Y(y)\,Z(z) \qquad (2)$$

and we will find the 3 functions.

This assumption works for many physical problems in E+M, thermodynamics, fluids, Quantum Mechanics, etc.

Now let's write eq. ① replacing ϕ by Eq. ②:

$$\nabla^2 \phi = 0 \qquad \phi(x,y,z) = XYZ \quad ②$$

then

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} = 0 \quad ③$$

Divide ③ by ②:

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-\beta^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{\gamma^2 = \alpha^2+\beta^2} = 0$$

$\rightarrow$ positive because $\phi(x,y,a) \neq 0$.

Now we have 3 equations:

$$\frac{\partial^2 X}{\partial x^2} = -\alpha^2 X \Rightarrow X(x) \propto \cos\alpha x,\ \sin\alpha x$$

or

$$e^{i\alpha x} \text{ or } e^{-i\alpha x}$$

$$\frac{\partial^2 Y}{\partial y^2} = -\beta^2 Y \Rightarrow Y(y) \propto \cos\beta y,\ \sin\beta y$$

or

$$e^{\pm i\beta y}$$

and

$$\frac{\partial^2 Z}{\partial z^2} = +\gamma^2 Z \Rightarrow Z(z) \propto \sinh\gamma z,\ \cosh\gamma z$$

or

$$e^{\pm\gamma z}$$

Then we see that

$$\phi(x,y,z) = \sum_{\alpha,\beta} \left(A_\alpha \cos\alpha x + B_\alpha \sin\alpha x\right) \times$$

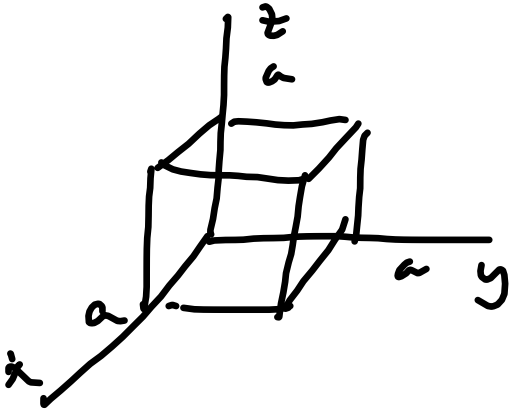
$$\times \left(A_\beta \cos\beta y + B_\beta \sin\beta y\right) \times$$

$$\left(A_{\alpha\beta} \cosh\gamma z + B_{\alpha\beta} \sinh\gamma z\right)$$

with $\gamma = \sqrt{\alpha^2+\beta^2}$

The values of A_i, B_j and α, β are obtained from the b.c.:

In our example:



$$\phi(x,y,z) \propto \sin\alpha x \sin\beta y \sinh\gamma z$$

$$\boxed{\phi(x,y,z) = \sum_{\alpha,\beta} C_{\alpha,\beta} \sin\alpha x \sin\beta y \sinh\gamma z}$$

$\cos\alpha x$ cannot be in the sol'n to a because $\phi(0,y,z)=0$.
$\cos\beta y$ " " " " " " $\phi(x,0,z)=0$
$\cosh\gamma z$ " " " " " $\phi(x,y,0)=0$

I used 3 b.c. to remove 3 sets of constants.

Now let's use the remaining 3 b.c.

$\phi(a, y, z) = 0$

then $\sin \alpha x \big|_{x=a} = \sin \alpha a = 0$

then $\alpha = \frac{\pi n}{a}$ $n = 1, 2, 3, \ldots$

$\alpha_n = \frac{\pi n}{a}$ $1, 2, 3, \ldots$

Also

$\phi(x, a, z) = 0$
then $\sin \beta y \big|_{y=a} = \sin \beta a = 0 \Rightarrow \beta_m = \frac{\pi m}{a}$ $m = 1, 2, 3, \ldots$

Then I have that

$$\phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{n,m} \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a} \times$$

$$\sinh \pi \frac{(n^2+m^2)^{1/2}}{a} z$$

Now we use the last b. c. to find $A_{n,m}$:

$$\phi(x,y,z=a) = V(x,y)$$

$$V(x,y) = \sum_{n,m=1}^{\infty} A_{n,m} \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a} \sinh \pi (n^2+m^2)^{1/2}$$

Since $\left\{ \sin \frac{n\pi x}{a} \right\}$ is an orthogonal set in $(0,a)$ for x

and $\left\{ \sin \frac{m\pi y}{a} \right\}$ " " " " " $(0,a)$ for y

We can use orthogonality to find $A_{n,m}$:

$$\int_0^a dx \int_0^a dy \; V(x,y) \sin \frac{n'\pi x}{a} \sin \frac{m'\pi y}{a} =$$

$$= \sum_{n,m} A_{nm} \sinh \pi (n^2+m^2)^{1/2} \underbrace{\int_0^a \sin \frac{n'\pi x}{a} \sin \frac{n\pi x}{a}\, dx}_{\frac{a}{2}\delta_{n,n'}}$$

$$\times \underbrace{\int_0^a \sin \frac{m'\pi y}{a} \sin \frac{m\pi y}{a}\, dy}_{} \rightarrow \frac{a}{2}\delta_{m,m'}$$

Then I obtain that

$$\int_0^a dx \int_0^a dy \; V(x,y) \sin\frac{n'\pi x}{a} \sin\frac{m'\pi y}{a} =$$

$$A_{n'm'} \frac{a^2}{4} \sinh \pi (n'^2 + m'^2)^{1/2}$$

Then:

$$\boxed{A_{nm} = \frac{4 \int_0^a dx \int_0^a dy \; V(x,y) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}}{a^2 \sinh \pi (n^2 + m^2)^{1/2}}}$$

If $V(x,y) = V$ then

$$A_{n,m} = \frac{4}{a^2} \frac{V}{\sinh \pi (n^2+m^2)^{1/2}} \left(-\frac{a}{n\pi}\right) \underbrace{\left[(-1)^n - 1\right]}_{\substack{0 \text{ for } n \text{ even} \\ -2 \text{ for } n \text{ odd}}} \left(-\frac{a}{m\pi}\right)$$

$$\underbrace{\left[(-1)^m - 1\right]}_{\substack{0 \text{ for } m \text{ even} \\ -2 \text{ for } m \text{ odd.}}}$$

$$A_{nm} = \frac{16}{nm\pi^2} \frac{V}{\sinh \pi (n^2+m^2)^{1/2}}$$ if n and m are odd

let's define $n = 2k+1$ $\quad m = 2j+1$

Then

$$\phi(x,y,z) = \frac{16V}{\pi^2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\sin\frac{(2k+1)\pi x}{a} \sin\frac{(2j+1)\pi y}{a}}{(2k+1)(2j+1)} \times$$

$$\times \frac{\sinh \frac{\pi}{a}\left[(2k+1)^2 + (2j+1)^2\right]^{1/2} z}{\sinh \pi \left[(2k+1)^2 + (2j+1)^2\right]^{1/2}}$$