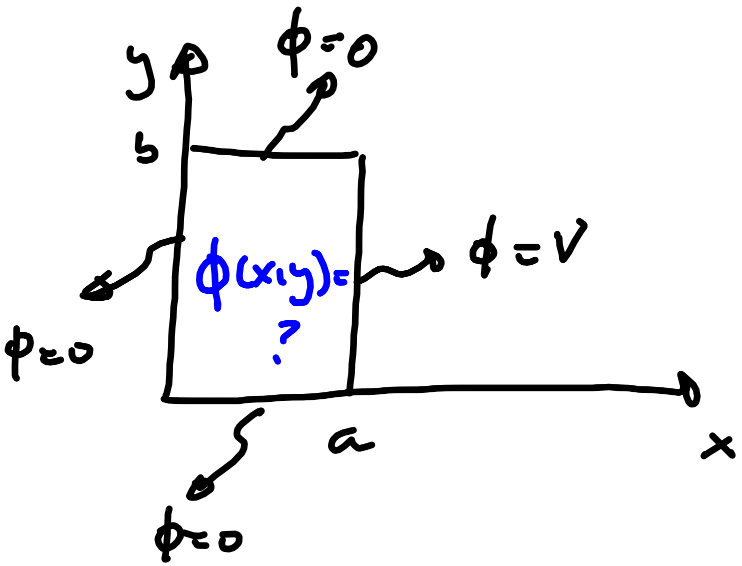
10/15

Separation of variables in cartesian coordinates.

Applications: 2D-example.



$$\nabla^2 \phi = 0 \quad \text{(no charge)}$$

$$\phi(x,y) = X(x)\,Y(y)$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{\alpha^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-\alpha^2} = 0$$

hyperbolic

periodic because of the b.c.'s.

Then we know that

$X(x) \propto e^{\pm \alpha x}$ or $\cosh \alpha x$, $\sinh \alpha x$

$Y(y) \propto e^{\pm i \alpha y}$ or $\sin \alpha y$, $\cos \alpha y$

Now we need to chose the specific forms of the solutions that satisfy the b.c..

$$\boxed{\phi(x,y) = \sum_{\alpha} A_\alpha \sin \alpha y \sinh \alpha x} = \sum_{n=1}^{\infty} A_n \sin \frac{\pi n y}{b} \sinh \frac{\pi n x}{b}$$

$\sinh \alpha x = 0$ for $x = 0$

Notice that $\sin(\alpha y) = 0$ for $y = 0$
and $\sin(\alpha y) = 0$ for $y = b$ if $\alpha = \frac{\pi n}{b}$

Now use the last b.c. $\phi(x=a, y) = V$ to find A_n:

$$V = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi a}{b} \sin\frac{n\pi y}{b}$$

Now multiply both sides by $\sin\frac{m\pi y}{b}$ and integrate over y:

$$V \underbrace{\int_0^b \sin\frac{m\pi y}{b}\, dy}_{-\frac{b}{m\pi}\cos\frac{m\pi y}{b}\Big|_0^b = -\frac{b}{m\pi}\left[(-1)^m - 1\right] = \begin{cases} \frac{2b}{m\pi} & m \text{ odd} \\ 0 & \text{for } m \text{ even} \end{cases}} = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi a}{b} \underbrace{\int_0^b \sin\frac{n\pi y}{b} \sin\frac{m\pi y}{b}\, dy}_{\frac{b}{2}\delta_{n,m}}$$

$$\frac{V\,2\cancel{b}}{m\pi} = A_m \frac{\cancel{b}}{2} \sinh\frac{m\pi a}{b} \quad \text{for } m \text{ odd}$$

$A_m = 0$ for m even

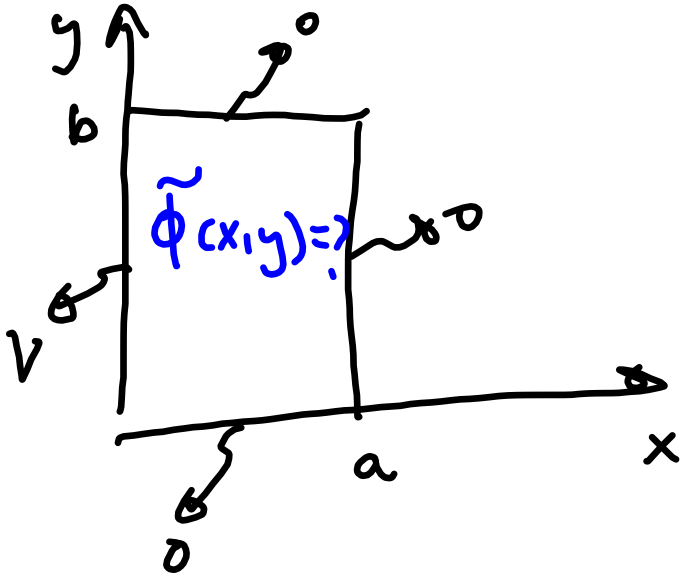
$$A_m = \frac{4V}{\pi m \sinh\frac{m\pi a}{b}} \quad \text{for } m \text{ odd.}$$

Now redefine $m = 2j+1$

Then:

$$\boxed{\phi(x,y) = \frac{4V}{\pi}\sum_{j=0}^{\infty} \frac{\sinh\frac{(2j+1)\pi x}{b}\,\sin\frac{(2j+1)\pi y}{b}}{(2j+1)\,\sinh\frac{(2j+1)\pi a}{b}}}$$

Now consider this problem:



$$\phi(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b} \times \left(A_n e^{\frac{\pi n x}{b}} + B_n e^{-\frac{\pi n x}{b}}\right) \qquad (I)$$

Notice: if you are astute you can use:

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{b} \times \sinh\frac{n\pi(a-x)}{b} \qquad (II)$$

In this case you obtain A_n from $\phi(x=0,y) = V$.

But let's try solution I:

$$\phi(x=a,y) = 0 = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b} \underbrace{\left(A_n e^{\frac{n\pi a}{b}} + B_n e^{-\frac{n\pi a}{b}}\right)}_{0}$$

$$A_n = -B_n e^{-\frac{2n\pi a}{b}}$$

Then:

$$\phi(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b} B_n \left(-e^{-\frac{2n\pi a}{b}} e^{\frac{n\pi x}{b}} + e^{-\frac{n\pi x}{b}}\right)$$

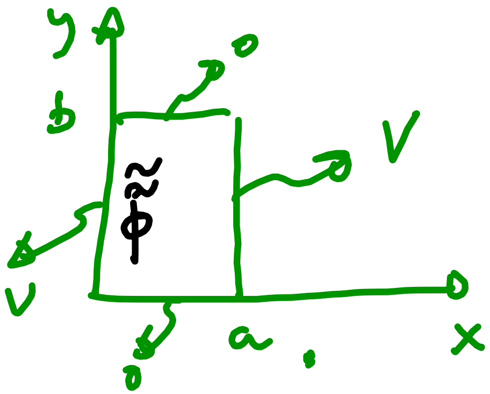
$$= \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi a}{b}} \sin\frac{n\pi y}{b} \underbrace{\left(-e^{-\frac{n\pi a}{b}} e^{\frac{n\pi x}{b}} + e^{-\frac{n\pi x}{b}} e^{\frac{n\pi a}{b}}\right)}_{2\sinh\frac{n\pi}{b}(a-x)}$$

$$= \sum_{n=1}^{\infty} 2B_n e^{-\frac{n\pi a}{b}} \sin\frac{n\pi y}{b} \sinh\frac{n\pi}{b}(a-x)$$

Then the last b.c. $\phi(x=0,y)=V$ allows us to obtain B_m (or A_m in sol. I). Then

$$\tilde{\phi}(x,y) = \frac{4V}{\pi}\sum_{j=0}^{\infty} \frac{\sinh\frac{(2j+1)\pi(a-x)}{b}\,\sin\frac{(2j+1)\pi y}{b}}{(2j+1)\sinh\frac{(2j+1)\pi a}{b}}$$

Now you can solve this problem:



$$\tilde{\tilde{\phi}}(x,y) = \phi(x,y) + \tilde{\phi}(x,y)$$

because of the principle of superposition.

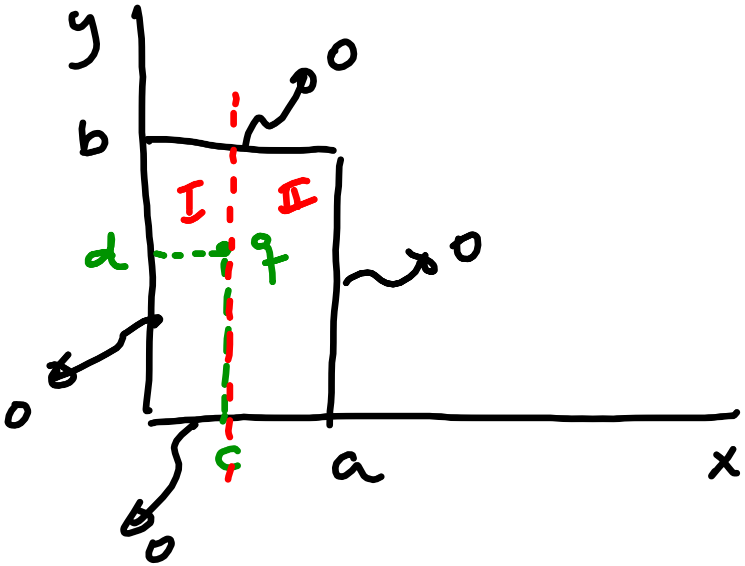
Warning: You cannot solve this problem in one single step!

Then:

$$\tilde{\phi}(x,y) = \frac{8V}{\pi} \sum_{j=0}^{\infty} \frac{\sinh (2j+1)\frac{\pi a}{2b} \cosh (2j+1)\frac{\pi}{b}(x-a/2)}{(2j+1)\sinh \frac{\pi(2j+1)a}{b}} \times$$

$$\times \sin \frac{(2j+1)\pi y}{b}$$

Now consider this problem!



Find $\phi(x,y)$ inside the box.

Now $\nabla^2\phi \neq 0$ in V

but I can partition V

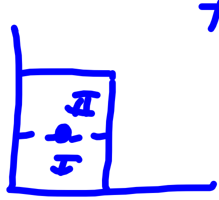
in regions I and II so

that $\nabla^2\phi^{I} = \nabla^2\phi^{II} = 0.$

Region I $0 \leq x \leq c$

Region II $c \leq x \leq a$

Notice:



this is also possible. It depends on the b.c. I will later use.

Let's propose our solutions!

$$\phi^{I}(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi x}{b}$$

$$\phi^{II}(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi (a-x)}{b}$$

We will find A_n and B_n from b.c. at:

$$\phi^{I}\Big|_{x=c} = \phi^{II}\Big|_{x=c}$$ because ϕ is continuous.

$$A_n \sinh\frac{n\pi c}{b} = B_n \sinh\frac{n\pi(a-c)}{b} \qquad (1)$$

Also $\bar{\nabla} \cdot \bar{E} = \rho/\varepsilon_0 \Rightarrow E_n^{II} - E_n^{I} = \rho/\varepsilon_0 = \frac{q\,\delta(y-d)}{\varepsilon_0}$

$\bar{E} = -\bar{\nabla}\phi$ and $E_n = -\frac{\partial \phi}{\partial \hat{n}}$

$\hat{n} = x \quad \therefore \quad E_n = -\frac{\partial \phi}{\partial x}$ in this case.

Then we have that

$$-\frac{\partial \phi^{II}}{\partial x}\bigg|_{x=c} + \frac{\partial \phi^{I}}{\partial x}\bigg|_{x=c} = \frac{q\,\delta(y-d)}{\varepsilon_0} \qquad (2)$$

From (1) I obtain:

$$A_n = B_n \frac{\sinh n\pi(a-c)/b}{\sinh n\pi a/b} \qquad (3)$$

From ② I get!

$$\sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{b}\right) \cosh \frac{n\pi (a-c)}{b} \sin \frac{n\pi y}{b} +$$

$$+ \sum_{n=1}^{\infty} A_n \frac{\pi n}{b} \cosh \frac{n\pi c}{b} \sin \frac{n\pi y}{b} = \frac{q}{\varepsilon_0} \delta(y-d)$$

replace using ③.

Then:

$$\sum_{n=1}^{\infty} B_n \frac{n\pi}{b} \sin \frac{n\pi y}{b} \left[\cosh \frac{n\pi(a-c)}{b} + \cosh \frac{n\pi c}{b} \frac{\sinh \frac{n\pi(a-c)}{b}}{\sinh \frac{n\pi c}{b}}\right]$$

$$= \frac{q}{\varepsilon_0} \delta(y-d) \qquad ④$$

Now multiply both sides of 4 by $\sin \frac{m\pi y}{b}$ and integrate over y between 0 and b:

From that you'll obtain

$$\boxed{B_n = \frac{2q}{\varepsilon_0 \pi} \sin\frac{n\pi d}{b} \frac{\sinh\frac{n\pi c}{b}}{\sinh\frac{n\pi a}{b}}} \qquad (5)$$

Plugging B_n in expression for A_n you get that:

$$\phi^{I}(x,y) = \frac{2q}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi d}{b}\,\frac{\sinh \pi n\frac{(a-c)}{b}}{\sinh\frac{\pi n a}{b}}\times$$

$$\times \sin\frac{n\pi y}{b}\,\sinh\frac{n\pi x}{b} \qquad \text{for } x \leq c.$$

$$\phi^{II}(x,y) = \frac{2q}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi d}{b}\,\frac{\sinh\frac{\pi n c}{b}}{\sinh\frac{\pi n a}{b}}\times$$

$$\times \sin\frac{n\pi y}{b}\,\sinh\frac{n\pi(a-x)}{b}. \qquad \text{for } c \leq x \leq a$$

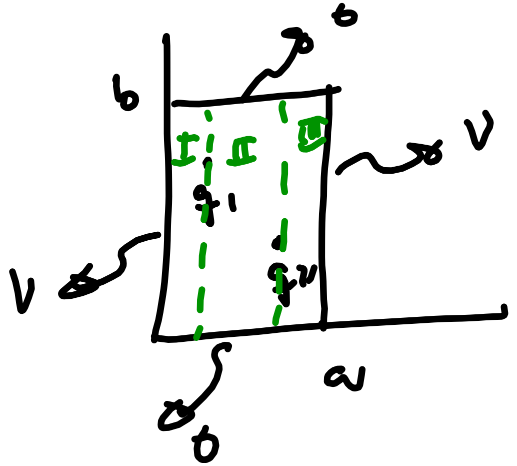
or

$$\phi(x,y) = \frac{2q}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi d}{b}\sin\frac{n\pi y}{b}\times$$

$$\times\frac{\sinh \pi n x_</b \times \sinh \pi n(a-x_>)/b}{\sinh \pi n a/b}$$

where $x_<$ ($x_>$) is the smaller (larger) between x and c.

Using this solutions you can solve problems like this:



ϕ^{I}, ϕ^{II}, and ϕ^{III} are obtained by using $x_<$ and $x_>$ for each charge.