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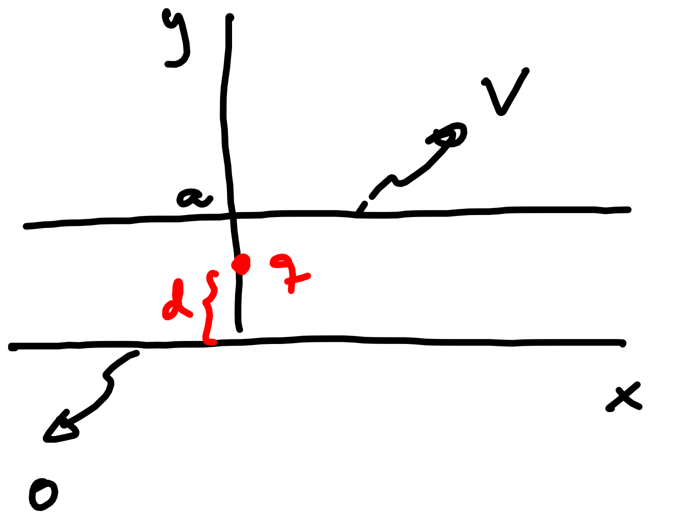
Hw #8: Problem 6.




Find $\phi(x,y)$ between the planes.

step ①: $\phi(x,y) \propto \sin\frac{n\pi y}{a}\, e^{\pm\frac{n\pi x}{a}}$

step ②: $\phi(x,y) = Y(y)$ with $X(x)=1$

$\nabla^2\phi = 0 \Rightarrow \frac{\partial^2 Y}{\partial y^2} = 0 \Rightarrow Y(y) = Ay$

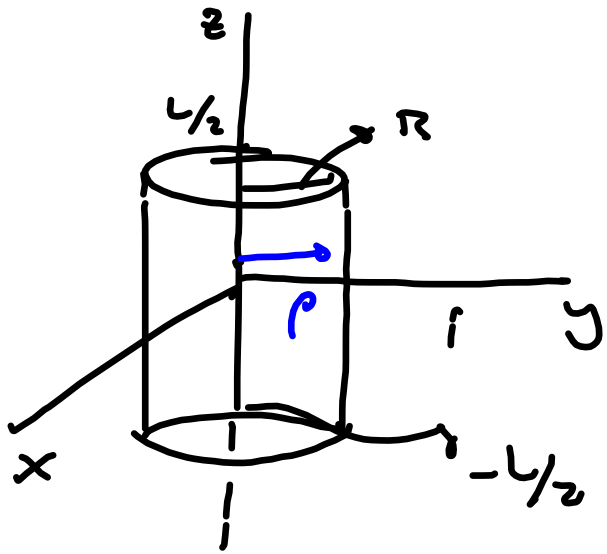
1) Find $\phi^{I}(x,y)$ due to $q$ with Dirichlet b.c.'s on the planes ($\phi=0$ on both planes).

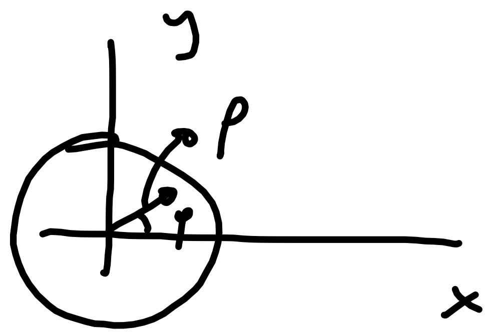
2) Find $\phi^{II}$ between the planes due to $\phi = V$ on top without the charge.

3) Add both solutions.

# Cylindrical coordinates.

If the boundary conditions are given on cylindrical surfaces we need to work in cylindrical coordinates: $(\rho, \varphi, z)$ (Ch. 7 and 14).




$\Phi(\rho, \varphi, z) = ?$ Inside or outside the cylinder.

1) Write $\nabla^2 \phi = 0$ in terms of $\rho, \varphi, z$:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (1)$$

2) Separation of variables:

$$\phi(\rho, \varphi, z) = P(\rho)\, Q(\varphi)\, Z(z) \qquad (2)$$

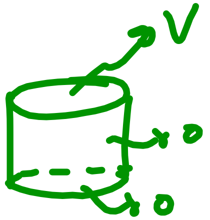
3) Plug (2) in (1):

$$QZ\,\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right)+\frac{PZ}{\rho^2}\frac{\partial^2 Q}{\partial\varphi^2}+PQ\,\frac{\partial^2 Z}{\partial z^2}=0$$

4) Divide by $\Phi$:

$$\underbrace{\frac{1}{P}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right)+\frac{1}{\rho^2 Q}\frac{\partial^2 Q}{\partial\varphi^2}}_{\substack{-k^2\\(k^2)}}+\underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{\substack{k^2\\(-k^2)}}=0$$

B.C. determine the sign of the constants that I chose:

if 

I want $e^{Ikz}$ solutions

if 

I want $e^{\pm ikz}$ solutions.

If we choose

$$\frac{d^2 Z}{dz^2} = k^2 Z \rightarrow \quad Z(z) \propto e^{\pm kz}$$

and

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho \frac{dP}{d\rho}\right) + \frac{1}{\rho^2 Q}\frac{\partial^2 Q}{\partial \varphi^2} = -k^2$$

5) Multiply by $\rho^2$:

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho \frac{dP}{d\rho}\right) + \underbrace{\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2}}_{-\nu^2} = -k^2 \rho^2$$

and

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+k^2\rho^2=\nu^2 \qquad (3)$$

Then

$$\Phi(\varphi)\propto e^{\pm i\nu\varphi}$$ periodic dependence on $\varphi$.

Now we need to find $P(\rho)$ solving (3).

$$\frac{\rho}{P}\frac{dP}{d\rho}+\frac{\rho^2}{P}\frac{d^2P}{d\rho^2}+(k^2\rho^2-\nu^2)=0 \qquad (4)$$

Multiply 4 by $\frac{P}{\rho^2}$:

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)P = 0 \qquad (5)$$

Now define $x = k\rho$

$dx = k\,d\rho$ then (5) becomes:

$$k^2\frac{d^2P}{dx^2} + \frac{k^2}{x}\frac{dP}{dx} + \left(k^2 - \frac{\nu^2 k^2}{x^2}\right)P = 0 \qquad (6)$$

÷ by $k^2$:

$$\boxed{\frac{d^2P}{dx^2} + \frac{1}{x}\frac{dP}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)P = 0} \quad (7)$$

Bessel's eq.

Solution : $P_\nu(x)$ is called the Bessel function of order $\nu$. (oscillatory behavior)

If we had chosen $-k^2$ instead of $k^2$ we would have obtained

$$\frac{d^2P}{dx^2} + \frac{1}{x}\frac{dP}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)P = 0$$

modified Bessel's equation (non oscillatory solutions)

# Frobenius Method (7.5 in book).

It solves linear, second order, homogeneous ordinary differential equations of the form:

$$y'' + P(x)\, y' + Q(x)\, y = 0 \quad (8) \qquad y' = \frac{dy}{dx}$$

Most general solution to (8) is: $\qquad y'' = \frac{d^2y}{dx^2}$

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$y_i$ are two independent solutions.

$C_i$ determined by b.c.

We assume that

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \qquad (9)$$

We plug (9) in (8) and we obtain $a_\lambda$ and $k$ from the resulting equations.

Example: we will apply the method to Bessel's eq.

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \qquad (10)$$

From (9):

$$y' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1} \qquad (11)$$

and

$$y'' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} \qquad (12)$$

Plug (9), (11) and (12) in (10) and look at the coefficients for each power of $x$:
The coefficients have to vanish independently.

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda} + \sum_{\lambda=0}^{\infty} (k+\lambda) x^{k+\lambda} a_\lambda +$$

$$+ \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda+2} - \sum_{\lambda=0}^{\infty} a_\lambda n^2 x^{k+\lambda} = 0$$

When $\lambda = 0$:

$$a_0 k(k-1) x^k + k x^k a_0 + \underbrace{a_0 x^{k+2}}_{} - a_0 n^2 x^k$$

this term has to be put together with other terms for $\lambda = ?$

The lowest power of $x$ is $x^k$. I want the coefficient of $x^k$ be 0 to satisfy the equation:

Then I ask that

$$a_0 k(k-1) + k a_0 - a_0 n^2 = 0$$

I obtain:

$$a_0 [k^2 \cancel{-k} \cancel{+k} - n^2] = 0$$

$$a_0 [k^2 - n^2] = 0$$

Since $a_0$ can be anything I set

$$k^2 = n^2 \Rightarrow \boxed{k = \pm n}$$ indicial equation.

Each value of $n$ $(\pm n)$ is going to give us the two independent solutions $y_1(x)$ and $y_2(x)$.

Now consider $k=1$. The lowest power of $x$ whose coefficient has to vanish is $x^{k+1}$ then we obtain:

$$a_1\left[(k+1)k + k + 1 - n^2\right] = 0$$

or

$$a_1 (k+1-n)(k+1+n) = 0$$

If $k = \pm n$ I need $a_1 = 0$ then all $a_{odd}$ are going to be 0.

If I ask $k = n-1$ or $k = -(n+1)$ then I need $a_0 = 0$ and all $a_{even} = 0$. The solutions obtain in each case will be the same.