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Last time:

We used Frobenius method to solve Bessel's eq!

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \quad (1)$$

- Find $y(x) = c_1 y_1(x) + c_2 y_2(x)$

- Assume that

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \quad (2)$$

• Plug ② in ③ and solve for a_λ and k:

$$\sum_{\lambda=0}^{\infty} a_\lambda \left\{ \left[(k+\lambda)(k+\lambda-1) + (k+\lambda) - n^2 \right] x^{k+\lambda} + x^{k+\lambda+2} \right\} = 0$$

• For $\lambda = 0$ we need:

$$a_0 \left(k(k-1) + k - n^2 \right) = 0 \Rightarrow \text{if } a_0 \neq 0 \text{ then } k = \pm n$$

• For $\lambda = 1$ we need:

$$a_1 \left((k+1)k + (k+1) - n^2 \right) = 0$$

$$a_1 (k+1+n)(k+1-n) = 0 \Rightarrow a_1 = 0 \text{ if } k = \pm n$$

if $n = \pm 1/2$ should do something else!

- Now consider $\lambda = j$ and let's obtain all the other a_j's:

consider $k = n$ ($k=-n$ will give us the other solution).

a_j will be in front of x^{j+n} but x^{j+n} will also be preceded by a_{j-2} and the sum of the 2 coefficients has to vanish.

$$a_j[(m+j)(m+j-1) + (m+j) - n^2] + a_{j-2} = 0$$

Define $i = j-2 \quad \therefore \quad j = i+2$

$$a_{i+2}[(m+i+2)(m+i+1) + (m+i+2) - n^2] + a_i = 0$$

Now set $i=j$ then

$$a_{j+2} \quad (j+2)(2n+j+2) + a_j = 0$$

then

$$a_{j+2} = \frac{-a_j}{(j+2)(2n+j+2)} \qquad (4)$$

then

$$a_2 = \frac{-a_0}{2(2n+2)} = \frac{-a_0\, n!}{4\,(n+1)!} = -\frac{a_0}{4(n+1)}$$

$$= \left(-\frac{a_0}{4(n+1)}\right)$$

$$a_4 = \frac{-a_2}{(2+2)(2n+2+2)} = \frac{-a_2}{4\underbrace{(2n+4)}_{8n+16}} = \frac{(-1)^2 a_0\, n!}{2^4\, 2!\,(n+2)!} =$$

$2+2$

$$= \frac{-a_0\, n!}{16\times 2\times 4(n+1)(n+2)!}$$

in general: $a_{2p} = \dfrac{(-1)^p a_0\, n!}{2^{2p}\, p!\, (n+p)!}$

Then

$$y_1(x) = a_0 x^n \left[1 - \frac{n!\, x^2}{2^2 (n+1)!} + \frac{n!\, x^4}{2^4\, 2!\, (n+2)!} + \cdots \right]$$

$$= a_0 \sum_{j=0}^{\infty} \frac{(-1)^j\, n!\, x^{n+2j}}{2^{2j}\, j!\, (n+j)!} =$$

$$= a_0 2^n n! \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^j}{j!\, (n+j)!} \left(\frac{x}{2}\right)^{n+2j}}_{J_n(x)}$$

valid for all $n \neq \pm 1/2$.

$J_n(x)$ Bessel function of order n.

To obtain $y_2(x)$ select $k=-n$ and follow a similar procedure.

However, if $n<0$ and an integer expression

(4) $$a_{j+2} = -\frac{a_j}{(j+2)(2n+j+2)}$$

diverges if $j=-2(n+1)$ then in this case it is found that

$$J_{-n}(x) = (-1)^n J_n(x)$$

for n integer

(Ch. 14).

Frobenius method is pretty useful but it may fail for some values of indices or for certain equations. One needs to check in each case. But we are going to apply it to other problems.

Laplace's equation in Spherical coordinates.

If the b.c. are given on spherical surfaces we need to use spherical coordinates;

$\therefore \bar{r} = (\rho, \theta, \varphi)$

$\nabla^2 \phi = 0 \qquad \phi = \phi(\rho, \theta, \varphi)$

$$\nabla^2 \phi = \frac{1}{\rho}\frac{\partial^2}{\partial \rho^2}(\rho\phi) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) +$$

$$+ \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0 \qquad (1)$$

Propose that

$$\phi(\rho,\theta,\varphi) = \frac{U(\rho)}{\rho} P(\theta) Q(\varphi) \qquad (2)$$

Plug ② in ① and multiply by $\frac{\rho^2 \sin^2\theta}{UPQ}$

$$\underbrace{\rho^2 \sin^2\theta\left[\frac{1}{U}\frac{\partial^2 U}{\partial \rho^2}+\frac{1}{P\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]}_{m^2}+$$

$$+\underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{-m^2}=0$$

Then

$$Q(\varphi)\propto e^{\pm i m\varphi} \qquad m=0,1,2,3,\ldots$$

Then we obtain

$$\frac{\rho^2 \sin^2\theta}{U}\frac{d^2U}{d\rho^2} + \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) - m^2 = 0$$

%. by $\sin^2\theta$:

$$\underbrace{\frac{\rho^2}{U}\frac{d^2U}{d\rho^2}}_{\ell(\ell+1)} + \underbrace{\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{-\ell(\ell+1)} = 0$$

Then:

$$\frac{d^2U}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2}U = 0 \Rightarrow U = A_\ell \rho^{\ell+1} + \frac{B_\ell}{\rho^\ell}$$

We are left with an equation for $P(\theta)$:

$$\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\ell(\ell+1)$$

Define $x = \cos\theta$ $\quad dx = -\sin\theta\, d\theta$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\frac{dP}{d\theta}\right) - \frac{Pm^2}{\sin^2\theta} + \ell(\ell+1)P = 0$$

We obtain: $1-x^2 = \sin^2\theta$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0$$

generalized Legendre equation.

Solutions are $P_\ell^m(x)$: generalized Legendre polynomials.

If $m=0$ then $Q(\varphi) = e^{\pm im\varphi} = 1$ then

$\phi(\rho,\theta,\varphi) = \phi(\rho,\theta)$ azimuthal symmetry!

Then if $m=0$ we have

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0 \qquad (1)$$

Legendre equation.

Solutions: $P_\ell(x)$: Legendre polynomials.

In this case then

$$\phi(\rho,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell \rho^\ell + \frac{B_\ell}{\rho^{\ell+1}}\right) P_\ell(\cos\theta)$$

Let's find $P_\ell(x)$:

What is the range of x?

Since $x = \cos\theta$ then $\theta : (0 - \pi)$

$x : (\cos 0 - \cos\pi)$

$\shortparallel$

$(1 - -1)$

x ranges between ± 1.

We will propose:

$$P(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \qquad 2$$

using Frobenius method.

Rewrite ①:

$$\frac{d^2P}{dx^2} - x^2\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \ell(\ell+1)P = 0 \quad ③$$

Plug ② in ③:

$$\sum_{\lambda=0}^{\infty}\Big\{(k+\lambda)(k+\lambda-1)\,a_\lambda\, x^{k+\lambda-2} - \Big[\underbrace{(k+\lambda)(k+\lambda+1)}_{(k+\lambda)(k+\lambda-1)+2(k+\lambda)} - \ell(\ell+1)\Big]a_\lambda\, x^{k+\lambda} = 0 \quad ④$$

• Now for $\lambda = 0$ we need that the coef. of x^{k-2} vanishes. This means.

$a_0 k (k-1) = 0$ indicial equation.

Then $k = 0$ or $k = 1$

• For $\lambda = 1$

$a_1 (k+1) k = 0$ then $k = -1$ or $k = 0$

if $a_1 \neq 0$.

or $a_1 = 0$ to satisfy

$k = 1$ or 0.