Last Time

10/29

Legendre equation:

$$\frac{d^2P}{dx^2} - x^2\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \ell(\ell+1)P = 0 \quad (1)$$

$$P(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \quad (2)$$

Plugging (2) in (1):

$$\sum_{\lambda=0}^{\infty}\Big\{(k+\lambda)(k+\lambda-1)a_\lambda x^{k+\lambda-2} -$$
$$- [(k+\lambda)(k+\lambda-1) - \ell(\ell+1)]a_\lambda x^{k+\lambda}\Big\} = 0$$

- We saw that for $\lambda = 0$ $k = 0$ or 1

and for $\lambda = 1$ we found $k = 0$ or -1.

- To make the coefficient of $x^{k+\lambda}$ vanish we need that:

$$(k+\lambda+2)(k+\lambda+2-1)\,a_{\lambda+2} = \left[(k+\lambda)(k+\lambda-1) - \ell(\ell+1)\right] a_\lambda$$

$\therefore$

$$a_{\lambda+2} = \frac{\left[(k+\lambda)(k+\lambda-1) - \ell(\ell+1)\right]}{(k+\lambda+2)(k+\lambda+1)}\, a_\lambda \qquad (3)$$

We can chose $a_0 \neq 0$ and $a_1 = 0$ or $a_0 = 0$ and $a_1 \neq 0$ - The results will be the same.

If $a_0 \neq 0$ and $a_1 = 0$ then $k = 0$ or 1.

If $k = 0$

$$P(x) = \sum_{j=0}^{\infty} a_{2j}\, x^{2j}$$

only even values of λ appear.
(even powers of x)

If $k = 1$

$$P(x) = \sum_{j=0}^{\infty} a_{2j}\, x^{2j+1}$$

only even values of λ appear
(odd powers of x).

Notice that $P(x=1)$ has to be finite.

If $x = 1$ and $a_{2j} \neq 0$ for all $j \Rightarrow P(x=1) \to \infty$.

We need to find solutions that are polynomials rather than ∞ series.

Consider $l=0$ and $k=1$ in (3)

(Notice that k, l and λ are always 0 or positive integers)

$$a_{\lambda+2} = \frac{(1+\lambda)\cancel{(\lambda+2)}}{\cancel{(\lambda+2)}(\lambda+3)} a_\lambda = \frac{(1+\lambda)}{(\lambda+3)} a_\lambda$$

Here $a_{\lambda+2} \neq 0 \ \forall \lambda$.

The series with odd powers of x is not a good solution for $l=0$.

Now lets consider $k=0$ and replace in ③:

$$a_{\lambda+2} = \frac{[\lambda(\lambda-1)]}{(\lambda+1)(\lambda+2)} a_\lambda$$

$a_0 \neq 0$ but $a_2 = 0$ and $a_{\lambda+2} = 0$
$\forall \ \lambda+2 \neq 0.$

$$P_{\ell=0}(x) = a_0 x^{\overset{\overset{0}{\shortparallel}}{k+0}} = a_0$$

. If you had selected $a_0 = 0$ and $a_1 \neq 0$
you would have obtained that
$P_{\ell=0}(x) = a_1$ same result.

For $\ell \neq 0$ you will find that

- $P_\ell(x)$ are polynomials of order ℓ.
- $P_\ell(x)$ is even if ℓ is even (all even powers of x)
- $P_\ell(x)$ is odd if ℓ is odd (all odd powers of x).

$P_\ell(x)$ Legendre polynomial of order ℓ.

with x^ℓ its highest power of x.

Normalization:

$$P_\ell(x) = 1 \quad \forall \ell.$$

$\therefore a_0 = 1$ (or $a_1 = 1$ if $a_0 = 0$).

With the normalization:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$\vdots$$

Properties of $P_\ell(x)$:

- Rodrigues' formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell$$

- Orthogonality: In the interval $[-1, 1]$:

$$\int_{-1}^{1} P_{\ell'}(x)\, P_\ell(x)\, dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

- Then any well behaved function of x in the interval $[-1, 1]$ can be written in terms of $P_\ell(x)$:

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x) \qquad \text{for} \quad -1 \le x \le 1.$$

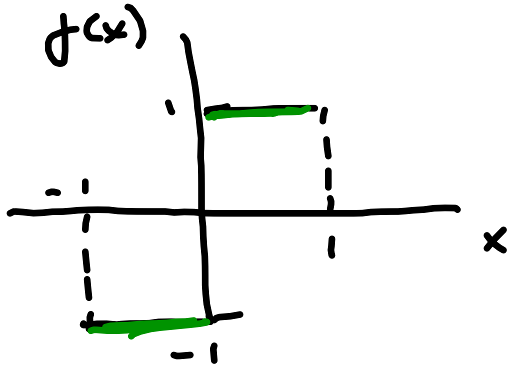
$$\int_{-1}^{1} f(x) P_{\ell'}(x)\, dx = \sum_{\ell=0}^{\infty} A_\ell \underbrace{\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$$\int_{-1}^{1} f(x) P_{\ell'}(x)\, dx = \frac{A_{\ell'}\, 2}{2\ell'+1}$$

if $\ell' = \ell$

$$\boxed{A_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} f(x) P_\ell(x)\, dx}$$

Example:



$$A_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} f(x)\, P_\ell(x)\, dx =$$

$$= \frac{2\ell+1}{2}\left[\int_{-1}^{0}(-1)\, P_\ell(x)\, dx + \int_{0}^{1} P_\ell(x)\, dx\right]$$

$A_\ell = 0$ if ℓ is even because $P_\ell(x) = P_\ell(-x)$

If ℓ is odd $P_\ell(x) = -P_\ell(-x)$ then

$$A_\ell = (2\ell+1)\int_0^1 P_\ell(x)\, dx = \frac{\left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}}(2\ell+1)(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

$m!! = m(m-2)(m-4)\cdots$

Now let's go back to our original problem:

$$\nabla^2 \phi = 0 \quad \text{with} \quad \phi(r,\theta,\varphi) \overset{m=0}{=} \phi(r,\theta) \propto \frac{U}{r} P(\theta)$$

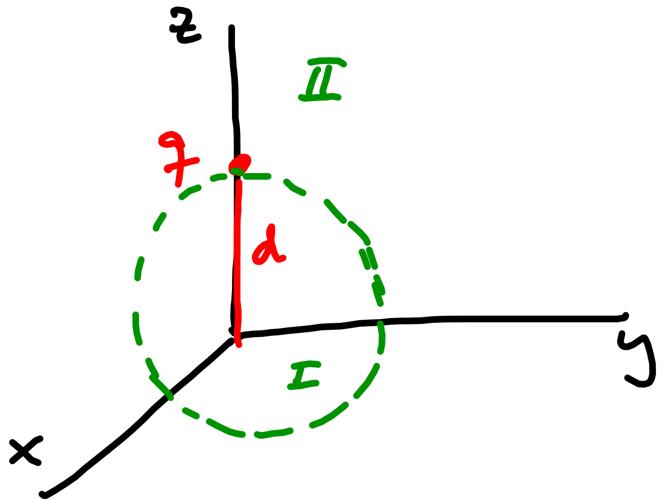
Then

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Most general solution for problems with b.c. on spherical surfaces and azimuthal symmetry.

A_ℓ and B_ℓ are given by the b.c.'s.

Example: Find the potential of a charge q in terms of $P_\ell(\cos\theta)$. This is the same as expanding $\frac{1}{|\bar{r}-\bar{r}'|}$ in terms of $P_\ell(\cos\theta)$.



$$\phi_q(r) = \frac{q}{4\pi\varepsilon_0 |\bar{r}-\bar{d}|}$$

I: $r \leq d$

$$\phi^{I}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

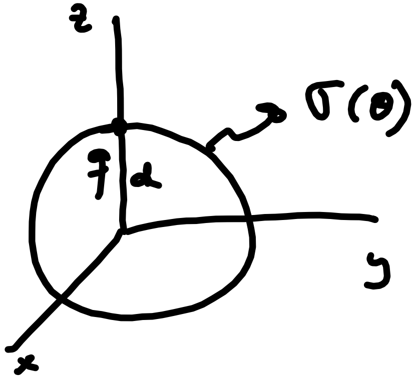
II: $r \geq d$

$$\phi^{II}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

To find A_ℓ and B_ℓ we use that:

$$\text{At } r=d \begin{cases} \phi^{I} = \phi^{II} \\ -\left.\dfrac{\partial \phi^{II}}{\partial r}\right|_{r=d} + \left.\dfrac{\partial \phi^{I}}{\partial r}\right|_{r=d} = \dfrac{\sigma}{\varepsilon_0} = \dfrac{q\,\delta(\cos\theta-1)}{2\pi d^2 \varepsilon_0} \end{cases}$$

what is σ?



$$\sigma = \frac{q\,\delta(\cos\theta-1)}{2\pi d^2}$$

Check:

$$d^2 \int_{-1}^{1} d(\cos\theta) \int_{0}^{2\pi} d\varphi\, \sigma(\theta) = \cancel{2\pi} \int_{-1}^{1} d(\cos\theta)\, \frac{q\,\delta(\cos\theta-1)}{\cancel{2\pi}\,\cancel{d^2}}\,\cancel{d^2} =$$

$$= q$$

From $\phi^{I} = \phi^{II}$ at $r = d$ we obtain

$$\sum_{\ell=0}^{\infty} A_\ell d^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{d^{\ell+1}} P_\ell(\cos\theta)$$

Then

$$A_\ell d^\ell = \frac{B_\ell}{d^{\ell+1}} \quad \text{or} \quad \boxed{A_\ell = \frac{B_\ell}{d^{2\ell+1}}}$$

Now

$$\phi^{I}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{d^{2\ell+1}} r^\ell P_\ell(\cos\theta)$$

$$\phi^{II}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$-\left.\frac{\partial \phi^{II}}{\partial r}\right|_{r=d} + \left.\frac{\partial \phi^{I}}{\partial r}\right|_{r=d} =$$

$$= \sum_{\ell=0}^{\infty} \left[\frac{(\ell+1)\, B_\ell}{d^{\ell+2}} + \frac{\ell\, B_\ell}{d^{\ell+2}} \right] P_\ell(\cos\theta) =$$

$$= \sum_{\ell=0}^{\infty} \frac{(2\ell+1)\, B_\ell}{d^{\ell+2}} P_\ell(\cos\theta) = \frac{q\, \delta(\cos\theta - 1)}{2\pi d^2 \varepsilon_0}$$

Use orthogonality of $P_\ell(\cos)$ to find B_ℓ :

$$\sum_{\ell=0}^{\infty} \frac{(2\ell+1) B_\ell}{d^{\ell+2}} \underbrace{\int_{-1}^{1} P_{\ell'}(\cos) P_\ell(\cos\theta)\, d(\cos\theta)}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}} = \frac{q}{2\pi d^2 \varepsilon_0} \overbrace{\int_{-1}^{1} P_{\ell'}(\cos\theta)\, \delta(\cos\theta - 1)\, d(\cos\theta)}^{P_{\ell'}(1)=1}$$

Then

$$\frac{\cancel{(2\ell'+1)}}{d^{\ell'+2}} B_{\ell'} \frac{2}{\cancel{(2\ell'+1)}} = \frac{q}{2\pi d^2 \varepsilon_0}$$

$$\boxed{B_\ell = \frac{q}{4\pi\varepsilon_0} d^\ell}$$

then

$$\boxed{A_\ell = \frac{B_\ell}{d^{2\ell+1}}} = \boxed{\frac{q}{4\pi\varepsilon_0 d^{\ell+1}}}$$

Then

$$\phi^{I}(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{d^{\ell+1}} P_\ell(\cos\theta)$$

$$\phi^{II}(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{d^{\ell}}{r^{\ell+1}} P_\ell(\cos\theta)$$

or

$$\phi(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

$r_<$ ($r_>$) the smaller (larger) between r and d.

If $q = 4\pi\varepsilon_0$ we get that $\phi(r,\theta) = \dfrac{1}{|r-d|}$

then

$$\boxed{\frac{1}{|r-d|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta)}$$