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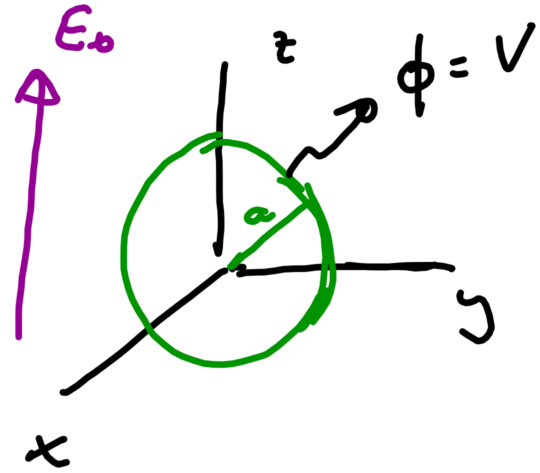
Application: 15.2.11.a



Find $\phi(r,\theta,\varphi)$ for $r \geq a$.

- Chose z axis parallel to $\bar{E}_0$ so that we have azimuthal symmetry.
- $\nabla^2 \phi = 0$ since there are not free charges.

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) - E_0 r P_1(\cos\theta)$$

$$\bar{E}_0 = E_0 \hat{z} \qquad \bar{E} = -\bar{\nabla}\phi_E \Rightarrow \phi_E = -E_0 z = -E_0 r \underbrace{\cos\theta}_{P_1(\cos\theta)}$$

To find B_ℓ I use that $\phi(r=a,\theta) = V$

$$V = \sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos\theta) - E_0 a P_1(\cos\theta)$$

$P_0(\cos\theta) = 1$

- $\ell = 0$ $\quad V = \dfrac{B_0}{a} \Rightarrow \boxed{B_0 = Va}$
- $\ell = 1$ $\quad 0 = \dfrac{B_1}{a^2} - E_0 a \Rightarrow \boxed{B_1 = E_0 a^3}$
- $\ell > 1$ $\quad \dfrac{B_\ell}{a^{\ell+1}} = 0 \quad \therefore \boxed{B_\ell = 0}$

Then we obtain:

$$\boxed{\phi(r,\theta) = \frac{aV}{r} + \frac{E_0 a^3}{r^2} \underbrace{P_1(\cos\theta)}_{\cos\theta} - E_0 r \underbrace{P_1(\cos\theta)}_{\cos\theta}}$$

$$\boxed{= \frac{aV}{r} + \frac{E_0 a^3}{r^2}\cos\theta - E_0 r \cos\theta}$$

Question: Find $\sigma(\theta)$ on the surface of the sphere.

$$-\frac{\partial \phi^{(r>a)}}{\partial r}\bigg|_{r=a} + \frac{\partial \phi^{(r\le a)}}{\partial r}\bigg|_{r=a} = \frac{\sigma}{\varepsilon_0}$$

Inside the sphere ϕ is constant then

$\frac{\partial \phi}{\partial r} = 0$ and we obtain

$$-\frac{\partial \phi^{(r>a)}}{\partial r}\bigg|_{r=a} = \frac{\sigma}{\varepsilon_0}$$

$$-\frac{\partial}{\partial r}\left[\frac{aV}{r} + \frac{E_0 a^3}{r^2}\cos\theta - E_0 r \cos\theta\right]\bigg|_{r=a} =$$

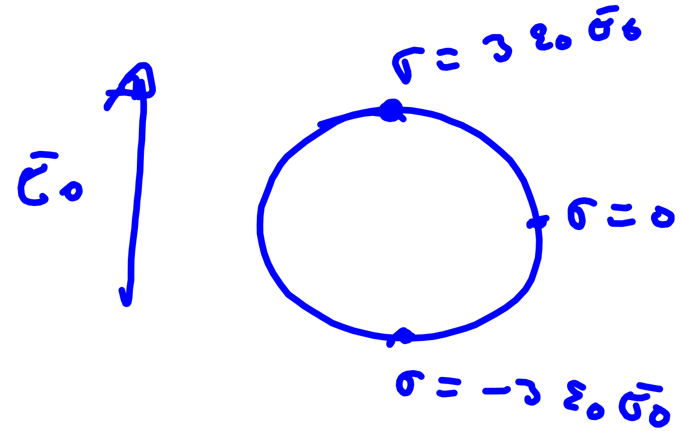
$$\frac{aV}{r^2} + \left(2\frac{E_0 a^3}{r^3} + E_0\right)\cos\theta\bigg|_{r=a} = \frac{aV}{a^2} + \underbrace{\left(2E_0 + E_0\right)}_{3E_0}\cos\theta = \frac{\sigma}{\varepsilon_0}$$

Then

$$\sigma(\theta) = 3\,\varepsilon_0 E_0 \cos\theta + \frac{V}{a}\varepsilon_0$$

If $V=0$ then

$$\boxed{\sigma(\theta) = 3\,\varepsilon_0 E_0 \cos\theta}$$



Problems without azimuthal symmetry.

Now m can take any value from $-\ell$ to ℓ.

The solutions to the differential eq. for $P(\cos\theta)$ is:

$P_\ell^m(\cos\theta)$ Associated Legendre polynomials.

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$$

Properties of $P_\ell^m(x)$:

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

$P_\ell^m(x)$ with fix m are orthogonal in $[-1,1]$:

$$\int_{-1}^{1} P_{\ell'}^m(x)\, P_\ell^m(x)\, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell,\ell'}$$

$-\ell \leq m \leq \ell$.

- Some problems (with magnetic fields) with azimuthal symmetry have $P_\ell^1(\cos\theta)$ as solutions.

Now the angular part of the potential will be given by

$$P_\ell^m(\cos\theta)\, e^{\pm i m\varphi} \propto \underbrace{Y_\ell^m(\theta,\varphi)}_{\text{spherical harmonic.}}$$

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

Spherical Harmonics : $Y_\ell^m(\theta, \varphi)$

Properties:

- ℓ ranges from 0 to ∞
- m ranges from $-\ell$ to ℓ.
- ℓ and m are integers.

$$Y_\ell^{-m}(\theta, \varphi) = (-1)^m Y_{\ell m}^*(\theta, \varphi)$$

$\{Y_\ell^m(\theta, \varphi)\}$ form an orthonormal basis in the intervals $(0 \leq \theta \leq \pi)$

$(0 \leq \varphi \leq 2\pi)$

Then:

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) =$$

$$= \delta_{\ell\ell'}\,\delta_{mm'}$$

Then any well behaved function $f(\theta,\varphi)$ can be written as:

$$f(\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m}\, Y_{\ell m}(\theta,\varphi)$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \; f(\theta,\varphi)\, Y^*_{\ell' m'}(\theta,\varphi) = A_{\ell' m'}$$

Then I can write the solution to

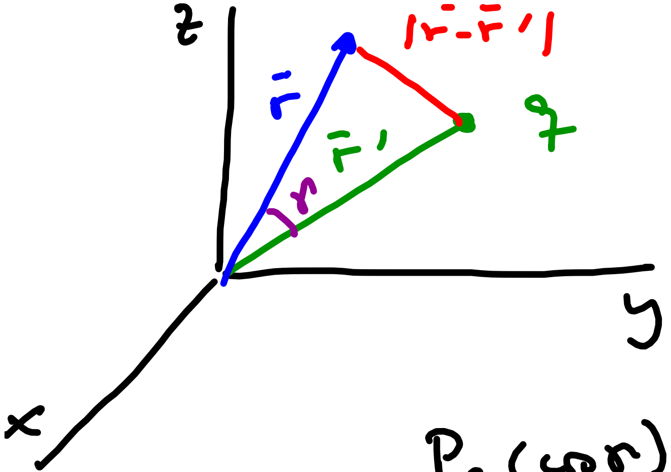
$$\nabla^2 \phi = 0$$

as

$$\boxed{\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell}^{m}(\theta,\varphi)}$$

Find the potential for a charge $q = 4\pi\varepsilon_0$ located at (x, y, z) in terms of $Y_{\ell m}(\theta, \varphi)$.

(it means to find $\frac{1}{|\bar{r} - \bar{r}'|}$ in terms of $Y_{\ell m}(\theta, \varphi)$:



We found that

$$\frac{1}{|\bar{r} - \bar{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\gamma) \quad (1)$$

Theorem of Addition of Spherical Harmonics:

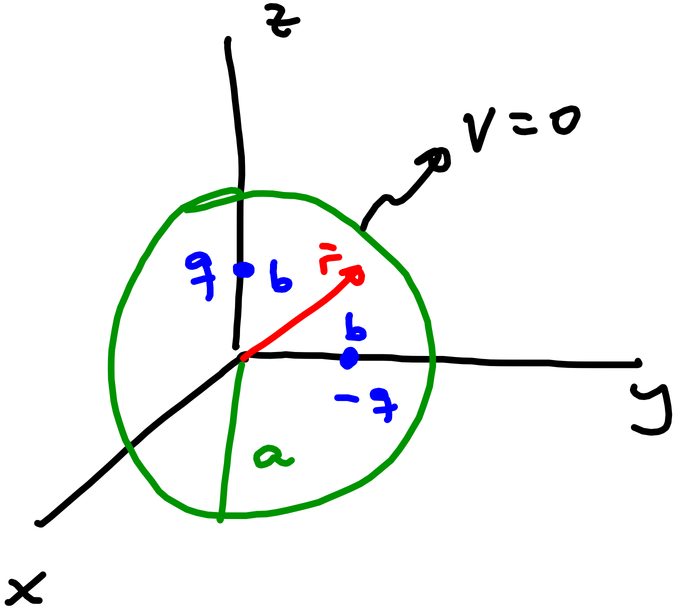
$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_\ell^{m}(\theta, \varphi)\, Y_\ell^{m*}(\theta', \varphi') \quad (2)$$

Replacing ② in ①:

$$\frac{1}{|\vec{r}-\vec{r}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_\ell^{m*}(\theta,\varphi)\, Y_\ell^{m}(\theta',\varphi')$$

$r_<$ ($r_>$) is the smaller (larger) between r and r'.

Example:



Find $\phi(r,\theta,\varphi)$ for $r \leq a$.

$$\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m} r^{\ell} Y_{\ell m}(\theta,\varphi)$$
$$+ \frac{q}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}(0,\varphi')$$
$$- \frac{q}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}\left(\frac{\pi}{2},\frac{\pi}{2}\right)$$

$r_<$ ($r_>$) is the smaller (larger) between r and b

To find $A_{\ell m}$ we use that $\phi(a, \theta, \varphi) = 0$.

$$0 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} a^{\ell} Y_{\ell m}(\theta, \varphi) +$$

$$+ \frac{q}{4\pi \varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{b^{\ell}}{a^{\ell+1}} Y_{\ell m}(\theta, \varphi) Y^{*}_{\ell m}(0, \varphi') -$$

$$- \frac{q}{4\pi \varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{b^{\ell}}{a^{\ell+1}} Y_{\ell m}(\theta, \varphi) Y^{*}_{\ell m}\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Now we use orthogonality to find $A_{\ell m}$. So we multiply by $Y^{*}_{\ell' m'}(\theta, \varphi)$ and integrate on θ and φ.

Then we obtain:

$$A_{\ell m}\, a^{\ell} + \frac{q}{4\pi\varepsilon_0}\,\frac{b^{\ell}}{a^{\ell+1}}\,\frac{1}{2\ell+1}\left[Y_\ell^{*m}(0,\varphi') - Y_\ell^{*m}\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right)\right]$$

$$= 0$$

Then

$$A_{\ell m} = \frac{q}{4\pi\varepsilon_0}\,\frac{b^{\ell}}{a^{2\ell+1}}\,\frac{1}{2\ell+1}\left[Y_\ell^{*m}\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right) - Y_\ell^{*m}(0,\varphi')\right]$$

$$Y_\ell^{*m}(0,\varphi') = \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^{m}(1)\; \underbrace{e^{-i\overset{0}{m}\varphi'}}_{1 \text{ because the potential does not depend on } \varphi}$$

$$= \sqrt{\frac{2\ell+1}{4\pi}}\; \underbrace{P_\ell(1)}_{1}$$