Levi - Civita tensor in Minkowski space 10/8

$$\varepsilon^{ijkl} = \begin{cases} 1 & \text{if } ijkl \text{ are in cyclic order} \\ -1 & \text{if } ijkl \text{ are not in cyclic order} \\ 0 & \text{if the indices are repeated.} \end{cases}$$

$$\varepsilon_{ijkl} = g_{ir}\, g_{js}\, g_{kt}\, g_{lu}\, \varepsilon^{rstu}$$

if $rstu = 0,1,2,3 \Rightarrow \varepsilon^{0123} = 1$

$$\varepsilon_{0123} = g_{00}\, g_{11}\, g_{22}\, g_{33}\, \varepsilon^{0123} = 1 \times (-1) \times (-1) \times (-1) \times 1 = -1$$

Then $\varepsilon_{0123} = -\varepsilon^{0123}$!

4- divergence:

$$\partial_\alpha A^\alpha = \partial_0 A^0 + \bar{\nabla}\cdot\bar{A} \qquad \text{scalar}$$

$$(\partial_0, \bar{\nabla}) \qquad (A^0, \bar{A})$$

Invariants:

consider $A^i = (dx^0, 0, 0, 0)$ $B^j = (0, dx^1, 0, 0)$

$C^k = (0, 0, dx^2, 0)$ $D^l = (0, 0, 0, dx^3)$

$$H^{ijkl} = A^i B^j C^k D^l$$

Let's construct:

$$\varepsilon_{ijkl} A^{ijkl} : \quad dx^0 dx^1 dx^2 dx^3 = d^4x$$

tensor of rank 0
(scalar and invariant)

volume element
in Minkowski
space

Warning!

$dx^1 dx^2 dx^3$ is NOT invariant.

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot (\underbrace{\rho \bar{v}}_{\bar{J}\text{ (density of current)}}) = 0 \quad (1)$$

In the book you'll see

$$j^\mu = \left(\rho, \frac{\rho \bar{v}}{c}\right) = \left(\rho, \frac{\bar{J}}{c}\right)$$

SI system.

Here I work in gaussian units because the expressions are simpler.

If we define:

$$J^\mu = (c\rho, \bar{J}) \quad \text{then (1) becomes:}$$

$$\frac{c}{c}\frac{\partial \rho}{\partial t} = \frac{\partial (c\rho)}{\partial (ct)} = \frac{\partial J^0}{\partial x^0} = \partial_0 J^0 \quad \text{and} \quad \bar{\nabla} \cdot \bar{J} = \partial_i J^i$$

$i = 1, 2, 3$

$$\therefore \boxed{\partial_\mu J^\mu = 0} \quad \text{continuity equation.}$$

Another 4-vector:

$$A^\mu = (\varphi, \bar{A})$$

$\bar{A}$: vector potential

φ : scalar potential

$$\begin{cases} \bar{B} = \bar{\nabla} \times \bar{A} & (1) \\ \bar{E} = -\frac{1}{c}\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\varphi & (2) \end{cases}$$

Book:

$$A^\mu = \varepsilon_0 (\varphi, c\bar{A})$$

and

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\varphi$$

SI system.

Both $\bar{A}$ and φ satisfy wave equations:

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = 4\pi\rho \\ \dfrac{1}{c^2}\dfrac{\partial^2 \bar{A}}{\partial t^2} - \nabla^2 \bar{A} = \dfrac{4\pi}{c}\bar{J} \end{cases}$$

if $J^\nu = (c\rho, \bar{J})$ and $A^\nu = (\varphi, \bar{A})$ we can write:

$$\boxed{\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu}$$

If $\nu = 0$

$$\partial_\mu \partial^\mu A^0 = \frac{4\pi}{c} J^0$$

$$\underbrace{\partial_\mu \partial^\mu}\varphi = \frac{4\pi}{\cancel{c}}\cancel{c}\rho$$

$$\frac{\partial^2}{\partial (ct)^2} - \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

Lorentz Gauge:

$\bar{\nabla}\cdot\bar{A}$ is arbitrary . . $\bar{\nabla}\cdot\bar{A}=0$ is Coulomb gauge useful in electrostatics.

For electrodynamics the Lorentz gauge is preferred:

$$\frac{1}{c}\frac{\partial\varphi}{\partial t}+\bar{\nabla}\cdot\bar{A}=0 \longrightarrow \boxed{\partial_\mu A^\mu=0}$$

Let's define:

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} = - F^{\nu\mu}$$ antisymmetric rank 2

$F^{\mu\nu}$ can be expressed in terms of $\bar{E}$ and $\bar{B}$.

Calculate F^{01}:

$$F^{01} = \partial^{0} A^{1} - \partial^{1} A^{0} = \frac{1}{c} \frac{\partial A_x}{\partial t} - \frac{\partial \varphi}{\partial x} = - E_x$$

from ①

Doing the same for all $F^{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Define

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$$ dual tensor of $F_{\gamma\delta}$

First let's find $F_{\alpha\beta}$:

$$F_{\alpha\beta} = g_{\alpha\gamma}\, g_{\beta\delta}\, F^{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{01} = \underbrace{g_{00}}_{1}\, \underbrace{g_{11}}_{-1}\, \underbrace{F^{01}}_{-E_x} = E_x = -F^{01}$$

then

$$\mathcal{F}^{01} = \frac{1}{2}\varepsilon^{01\gamma\delta} F_{\gamma\delta} = \frac{1}{2}\underbrace{\varepsilon^{0123}}_{1}\underbrace{F_{23}}_{-B_x} + \frac{1}{2}\underbrace{\varepsilon^{0132}}_{-1}\underbrace{F_{32}}_{B_x} =$$

$$= -B_x$$

Then

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

antisymmetric pseudotensor of rank 2.
(because of ε-ijk in it).

Now:

$$\left.\begin{aligned} \bar{\nabla}\cdot\bar{E} &= 4\pi\rho \\ \bar{\nabla}\times\bar{B} - \frac{1}{c}\frac{\partial\bar{E}}{\partial t} &= \frac{4\pi}{c}\bar{J} \end{aligned}\right\} \boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$$

inhomogeneous Maxwell's equations

$$\bar{\nabla}\cdot\bar{B}=0$$

$$\bar{\nabla}\times\bar{E}+\frac{1}{c}\frac{\partial\bar{B}}{\partial t}=0$$

$$\boxed{\partial_\alpha \mathcal{F}^{\alpha\beta}=0}$$

homogeneous Maxwell's equations

Also the above equations can be written as:

$$\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}+\partial_\gamma F_{\alpha\beta}=0$$

If $\alpha=1 \quad \beta=2 \quad \gamma=3$

$$\partial_1 F_{23}+\partial_2 F_{31}+\partial_3 F_{12}=0 \Rightarrow \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0$$

$$\therefore \bar{\nabla}\cdot\bar{B}=0$$

Differential Equations

In Physics differential equations rule.

$\nabla^2 \psi = 0$ Laplace eq. (homogeneous)

$\bar{F} = m \ddot{x}$ Newton

$\nabla^2 \psi = -\rho/\varepsilon_0$ Poisson

$\nabla^2 \psi + k^2 \psi = 0$ wave equation

$-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = i\hbar \frac{\partial \psi}{\partial t}$ Schrödinger's eq.

We are going to learn some techniques to solve this kind of equations:

1) Separation of variables, (homogeneous eq.s. but we'll consider inhomogeneities at the boundaries)

2) Green functions: (inhomogeneous problems)

3) Frobenius method (certain ordinary differential eq.)