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Last time:

$\vec{\nabla} = \partial_\mu = \frac{\partial}{\partial x^\mu}$ covariant derivative

and

$\nabla^2 = \partial_\mu \partial^\mu$ Laplacian.

with $\partial^\mu = \frac{\partial}{\partial x_\mu}$ contravariant derivative.

but we will learn about the metric tensor $g^{\mu\nu}$ then we will use that $\partial^\mu = g^{\mu\nu}\partial_\nu$.

Examples:

$\bar{E} = -\bar{\nabla}\phi$ and $\bar{\nabla}\cdot\bar{E} = 0$ if there are no charges.

$$0 = \bar{\nabla}\cdot\bar{E} = \bar{\nabla}\cdot(-\bar{\nabla}\phi) = -\nabla^2\phi = 0$$

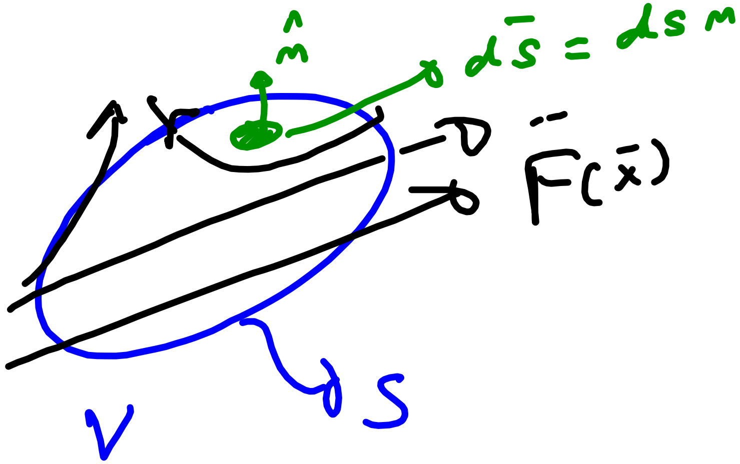
Then $\boxed{\nabla^2\phi = 0}$ if no charges

Laplace's equation.

We will learn techniques to solve it.

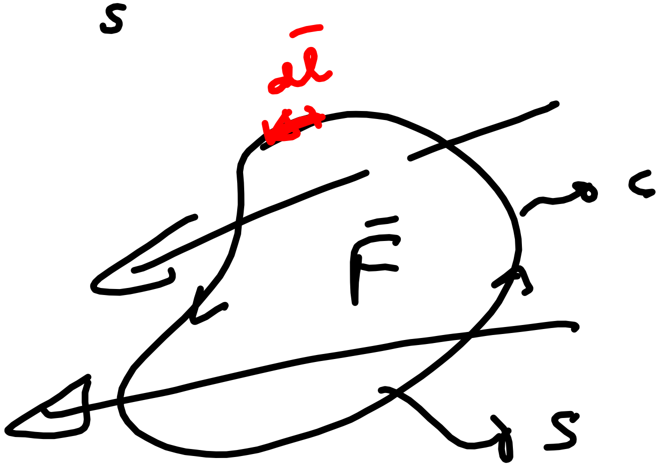
Divergence or Gauss' theorem:

$$\int_V \bar{\nabla} \cdot \bar{F} \, dV = \oint \bar{F} \cdot d\bar{s}$$



Stoke's Theorem:

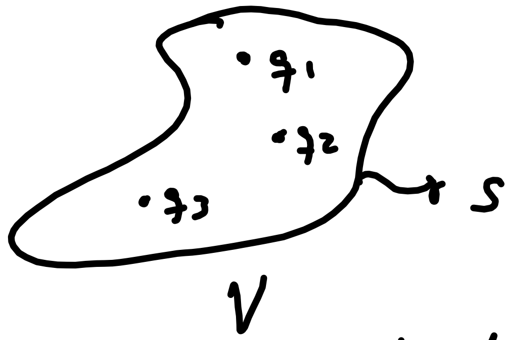
$$\int_S \bar{\nabla} \times \bar{F} \cdot \bar{ds} = \oint_C \bar{F} \cdot \bar{dl}$$



Example:

1) Gauss' law:

$$\oint_S \bar{E} \cdot d\bar{s} = \frac{q_{enclosed}}{\varepsilon_0}$$



Always valid but useful in symmetric situations.

$$\oint_S \bar{E} \cdot d\bar{s} \underset{\text{divergence theorem}}{=} \int_V \underbrace{\bar{\nabla} \cdot \bar{E}}_{\rho/\varepsilon_0 \text{ (Maxwell's)}} dV =$$

$$\int \rho \, dV = q$$

$$= \int_V \frac{\rho}{\varepsilon_0} dV = \frac{q_{enclosed}}{\varepsilon_0}$$

2) Poisson's equation:

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\varepsilon_0}$$ (Maxwell eq.)

also $$\bar{E} = -\bar{\nabla} \varphi$$ (electrostatics)

$$\therefore \quad \bar{\nabla} \cdot \bar{E} = -\nabla^2 \varphi = \frac{\rho}{\varepsilon_0} \Rightarrow \boxed{\nabla^2 \varphi = -\frac{\rho}{\varepsilon_0}}$$

Poisson's eq.
(inhomogeneous differential equation)

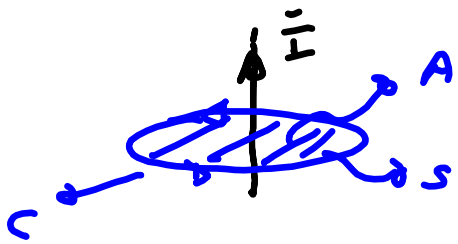
3) Ampère's law: $\bar{\nabla} \times \bar{H} = \bar{J}$ (Maxwell's eq.).

$$\int_S \bar{\nabla} \times \bar{H} \cdot d\bar{S} = \int_S \bar{J} \cdot d\bar{S} = \bar{I}_{\text{through the surface.}}$$

|| Stokes theorem

$$\oint_C \bar{H} \cdot d\bar{\ell}$$

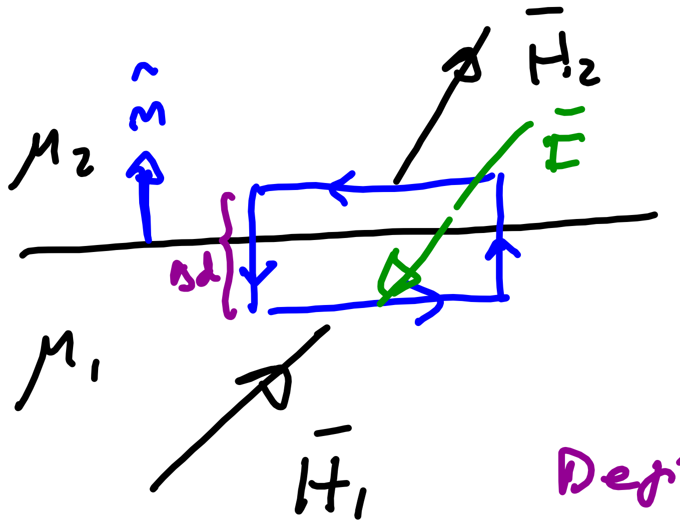
$\bar{J} = \dfrac{\bar{I}}{A}$ density of current $\bar{I}$ throug area A.



Then

$$\oint_C \bar{H} \cdot d\bar{\ell} = \bar{I}_{\text{through the area}}$$

In HW #3 you will demonstrate the boundary conditions for the $\bar{H}$-field at an interface:



What happens with tangential component of $\bar{H}$ across the interface?

$$\bar{J} = \frac{\bar{I}}{A}$$

Define:

$$\bar{K} = \lim_{\substack{\Delta d \to 0 \\ J \to \infty}} \bar{J}\,\Delta d$$

linear density of current

Similar to

$$\bar{p} = \lim_{\substack{d \to 0 \\ q \to \infty}} q d$$

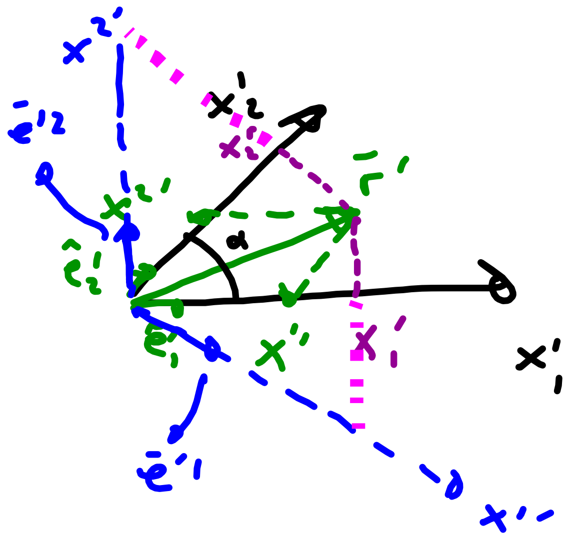
point like dipole

$q \bullet \underset{d}{\frown} \bullet -q$

Also in HW #3:

You are going to show that in the oblique system:

$\bar{r}' = x'^{i} \hat{e}_i$ in real space.



and

$\bar{r}' = x'_i \bar{e}^{i'}$ in dual space

You will find out that

$|\bar{e}^{i'}| = \operatorname{cosec}\alpha$ then $|\bar{e}^{i'}| = 1$ only if $\alpha = \pi/2$.

Dirac Delta "Function".

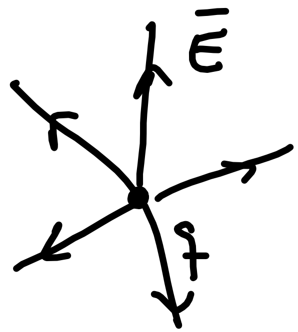
If $V = V(r) \Rightarrow$ equipotentials are spheres.

$r = (x^2+y^2+z^2)^{1/2}$

$\bar{\nabla} V = \frac{\partial V}{\partial r}\hat{r}$ gradient is $\perp$ to equipotentials.

Consider a point charge $q = 4\pi\varepsilon_0$

$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\varepsilon_0}$ $\bar{E} = -\bar{\nabla}\phi$



$\bar{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{\hat{r}}{r^2}$ and $\phi = \frac{1}{r}$

q @ origin.

Now let's use Gauss' theorem for our charge $q = 4\pi\varepsilon_0$ at the origin:

$$\oint_S \bar{E}\cdot d\bar{S} = \int_V \bar{\nabla}\cdot\bar{E}\, dV = \int_V \frac{\rho(\bar{x})}{\varepsilon_0}\, dV = \frac{q_{enc}}{\varepsilon_0} \begin{cases} 4\pi & \text{if origin in V.} \\ \text{or} & \\ 0 & \text{if origin not in V.} \end{cases} \qquad (1)$$

$$\bar{E} = -\nabla\varphi = -\nabla\left(\frac{1}{r}\right) = \frac{\hat{r}}{r^2}$$

$$\oint_S \bar{E}\cdot d\bar{S} = \oint_S \frac{\hat{r}}{r^2}\cdot d\bar{S} = \int_V \bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) dV =$$

$$= \int_V \bar{\nabla}\cdot\left(-\bar{\nabla}\left(\frac{1}{r}\right)\right) dV = -\int_V \nabla^2\left(\frac{1}{r}\right) dV$$

We define $\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\bar{r}) = -4\pi\,\delta(x)\,\delta(y)\,\delta(z) \qquad (2)$

Comparing (2) with (1) we see that

$$\int_V \delta(\bar{r})\, dV = \begin{cases} 1 & \text{if } \bar{r}=0 \in V \\ 0 & \text{if } \bar{r}=0 \notin V \end{cases}$$

Also notice that

$$\nabla^2\left(\frac{1}{r}\right) = \begin{cases} 0 & \text{if } r \neq 0 \\ \text{diverges} & \text{if } r=0. \end{cases}$$

Properties of $\delta(x)$:

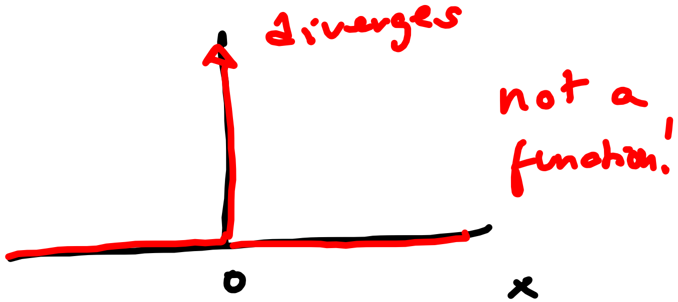
$$\delta(x) = 0 \quad \text{if } x \neq 0$$

$$f(0) = \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$$

If $f(x) = 1$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = \int_{-\infty}^{\infty} \delta(x)\,dx = f(0) = 1$$

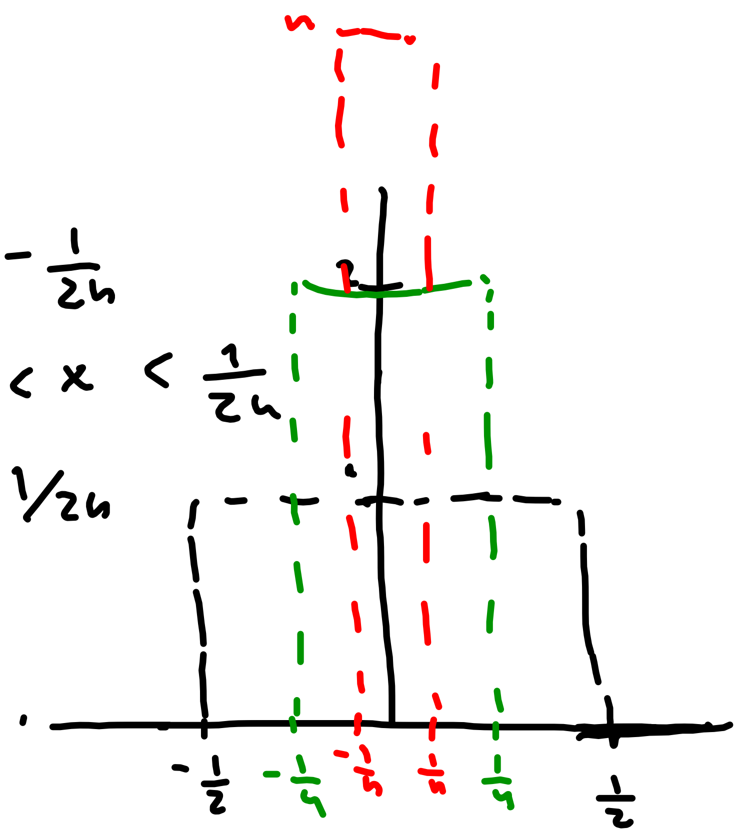
Then $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.



The $\delta(x)$ is called a "distribution". It can be obtained as the limit of a sequence of functions called "support" functions but it is NOT a function itself.

Ex:

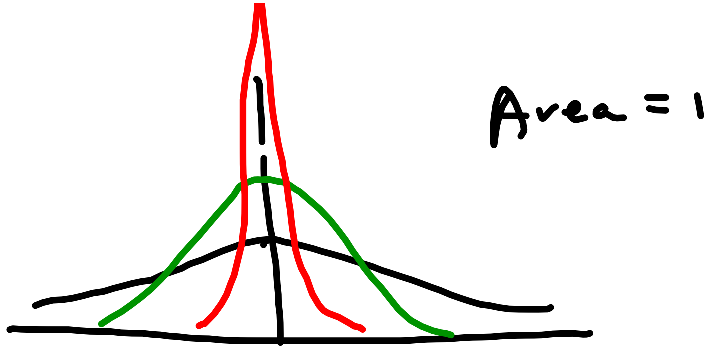
$$\delta_n(x) = \begin{cases} 0 & \text{if } x < -\frac{1}{2n} \\ n & \text{if } -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & \text{if } x > 1/2n \end{cases}$$



Also:

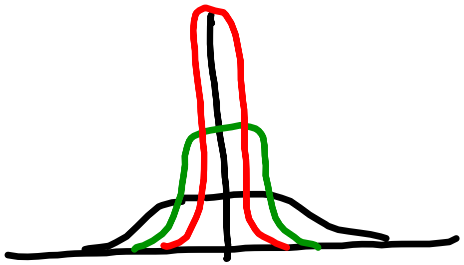
* $$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$$

Gaussians



* Lorentzians:

$$\delta_n(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2}$$



* $$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt} dt$$