Delta Function (cont.)

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Last time we found $\delta_n(x)$ functions such that

$$\lim_{n\to\infty} \delta_n(x) \quad \text{does not exist.}$$

But

$$\lim_{n\to\infty} \int d_n(x) f(x)\, dx = f(0)$$

if $\int_{-\infty}^{\infty} \delta_n(x)\, dx = 1$ for all n then $\delta_n(x)$ are normalized.

Now we define

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = \underbrace{\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x)\, f(x)\, dx = f(0)}_{\text{this can be prooved.}}$$

Example:

$$\lim_{n\to\infty} \int_{-\frac{1}{2n}}^{1/2n} n\, f(x)\, dx = \lim_{n\to\infty} n \overbrace{\int_{-1/2n}^{1/2n} f(x)\, dx}^{f(0)\Delta = f(0)\frac{1}{n}} =$$

$$= \lim_{n\to\infty} n \frac{1}{n} f(0) = f(0).$$

Properties of the delta:

- $$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = \int_{-\infty}^{\infty} f(y+a)\,\delta(y)\,dy = f(a)$$

  $y = x-a$

  $dy = dx$

- $$\int_{-\infty}^{\infty} f(x)\,\delta(g(x))\,dx = ?$$

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$$

$g(x_i)=0$

x_i are zeroes of $g(x)$

$g'(x_i) = \left.\frac{\partial g}{\partial x}\right|_{x=x_i}$

Example:

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx = \int_{-\infty}^{\infty} f(x)\,\frac{\delta(x)}{|a|}\,dx = \frac{f(0)}{|a|}$$

$g(x) = ax \qquad x_1 = 0$

$g'(x) = a$

- $$\int_{-\infty}^{\infty} f(x)\,\delta'(x-x')\,dx = \underbrace{f(x)\,\delta(x-x')}_{0}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x-x)\,dx =$$

integrating by parts

because $\delta(x-x')=0$ if $x \neq x'$

$$\int_a^b u\,v'\,dx = uv\Big|_a^b - \int_a^b u'\,v\,dx$$

$$\boxed{= -f'(x')}$$

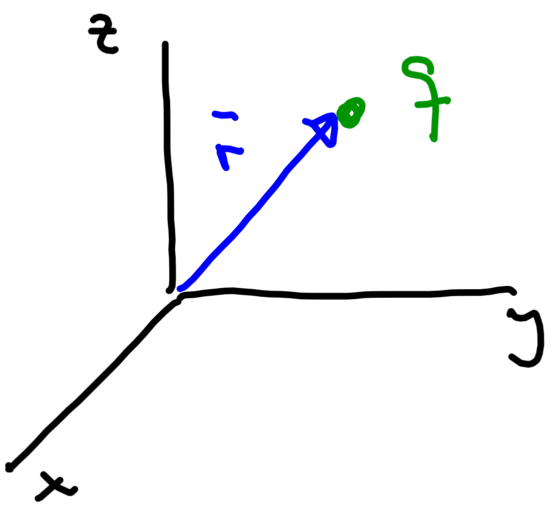
Delta function in 3 D:

In cartesian coordinates:

$$\delta(\bar{x}-\bar{X}) = \delta(x_1 - X_1)\,\delta(x_2 - X_2)\,\delta(x_3 - X_3)$$

$$\int_V \delta(\bar{x}-\bar{X})\,d^3x = \begin{cases} 1 & \text{if } \bar{X} \text{ is inside } V \\ 0 & \text{if } \bar{X} \text{ is not in } V. \end{cases}$$

- Mathematically $\delta(\bar{x}-\bar{X})$ has meaning only in an integral, but physicists use $\delta(\bar{x}-\bar{X})$ to denote singularities:



$$\rho(\bar{x}) = ?$$

$$q = \int_V \rho(\bar{x})\, d^3x = \int_V q\, \delta(\bar{x}-\bar{r})\, d^3x$$

then

$$\rho(\bar{x}) = q\, \delta(\bar{x}-\bar{r}) = q\, \delta(x-r_x)\, \delta(y-r_y)\, \delta(z-r_z)$$

in cartesian

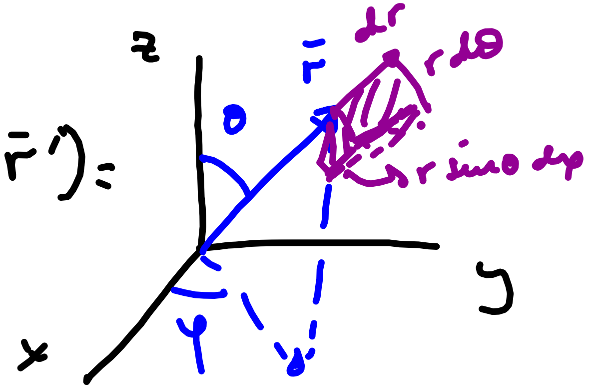
What happens if we want to express $\rho(\bar{x})$ in spherical coordinates?

Now I'll go from (x, y, z) to (r, θ, φ):

I want that $\int_V \rho(\bar{r})\, d^3x = q$ and

$$\int_{\text{all space}} \delta(\bar{r}-\bar{r}')\, d^3x = 1$$

$$1 = \int_0^\infty \int_0^\pi \int_0^{2\pi} C \underbrace{dr\, d\varphi\, d\theta\, r^2 \sin\theta}_{dV}\, \delta(\bar{r}-\bar{r}') =$$



$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} C\, dr\, d\varphi\, d\theta\, r^2 \sin\theta\, \delta(r-r')\, \delta(\theta-\theta')\, \delta(\varphi-\varphi') =$$

$$1 = \int_0^{\infty} \underbrace{C\, r^2 dr\ \delta(r - r')}_{C\, r'^2} \int_0^{\pi} \underbrace{\sin\theta\ \delta(\theta - \theta')\, d\theta}_{\sin\theta'}$$

$$\int_0^{2\pi} \underbrace{\delta(\varphi - \varphi')\, d\varphi}_{1} \Rightarrow$$

$$C = \frac{1}{r'^2 \sin\theta'}$$ then

$$\delta(\bar{r} - \bar{r}') = \frac{\delta(r - r')\,\delta(\theta - \theta')\,\delta(\varphi - \varphi')}{r'^2 \sin\theta'} = \frac{\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')}{r^2 \sin\theta}$$

In general:

$$\delta(\bar{x}-\bar{x}') = \frac{\delta(\xi_1-\xi_1')\,\delta(\xi_2-\xi_2')\,\delta(\xi_3-\xi_3')}{|J(x_i,\xi_i)|}$$

Jacobian

Cartesian

new coordinates

$$J(x_i,\xi_i) = \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix}$$

if $(\xi_1,\xi_2,\xi_3) = (r,\theta,\varphi)$ then

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$$J(x_i, r,\theta,\varphi) = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} =$$

$$= r^2\sin\theta$$

Note: if you use $\bar{r} = (r, \cos\theta, \varphi)$ then

$$\delta(\bar{r}-\bar{r}') = \frac{\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\varphi-\varphi')}{r^2}$$

$$d\cos\theta = -\sin\theta\, d\theta$$

Other uses of the δ:

- We can define an operator

$$\mathcal{L}(x_0) = \int dx \ \delta(x-x_0) \quad \text{so that}$$

$$\mathcal{L}(x_0) f(x) = \int dx \ f(x) \ \delta(x-x_0) = f(x_0)$$

- Map functions

$$\int_{-\infty}^{\infty} f(x) \ \delta(x-x_0) \ dx = f(x_0)$$

Expansion of $\delta(x-x')$ in terms of orthogonal functions: (see 5.1)

$\{\varphi_n(x)\}$ are a basis of orthogonal functions if:

$$\int_a^b \varphi_m^*(x)\,\varphi_n(x)\,dx = \delta_{m,n}$$

these are orthonormal functions.

if $\int_a^b \varphi_m^* \varphi_n dx \propto \delta_{m,n}$ they are orthogonal.

Any well behaved function $f(x)$ can be expanded in terms of $\{\varphi_n(x)\}$:

$$f(x) = \sum_{n=0}^{\infty} b_n \varphi_n(x)$$

b_n can be found due to the properties of $\{\varphi_n(x)\}$:

$$\int_a^b \varphi_m^*(x) f(x)\, dx = \underbrace{\sum_{n=0}^{\infty} b_n \underbrace{\int_a^b \varphi_m^*(x) \varphi_n(x)\, dx}_{\delta_{m,n}}}_{b_m}$$

Then

$$\boxed{b_m = \int_a^b f(x) \varphi_m^*(x)\, dx}$$

Now consider $f(x) = \delta(x-t)$

$$\delta(x-t) = \sum_{n=0}^{\infty} a_n \varphi_n(x)$$

$$a_m = \int_a^b \delta(x-t)\, \varphi_m^*(x)\, dx = \varphi_m^*(t)$$

$$\boxed{\delta(x-t) = \sum_{m=0}^{\infty} \varphi_n^*(t)\, \varphi_n(x)}$$

Example:

$$\{\varphi_n(x)\} = \left\{ \sqrt{\frac{2}{a}} \cos \frac{2n\pi x}{a} \right\}$$ orthonormal in $\left(-\frac{a}{2}, \frac{a}{2}\right)$

We can see that

$$\frac{2}{a} \int_{-a/2}^{a/2} \cos \frac{2n\pi x}{a} \cos \frac{2m\pi x}{a}\, dx = \underbrace{\delta_{n,m}}_{\text{Kronecker's } \delta}$$

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Then:

$$\delta(x-t) = \sum_{m=0}^{\infty} a_m \cos \frac{2\pi m x}{a} \sqrt{\frac{2}{a}}$$

$$a_n = \sqrt{\frac{2}{a}} \int_{-a/2}^{a/2} \cos \frac{2n\pi x}{a}\, \delta(x-t)\, dx = \sqrt{\frac{2}{a}} \cos \frac{2n\pi t}{a}$$

Then

$$\delta(x-t) = \frac{2}{a} \sum_{m=0}^{\infty} \cos\frac{2n\pi t}{a} \cos\frac{2n\pi x}{a}$$

Now let's go back to tensors (Ch. 4 in book).

- Up to now I talked about "tensors" but as a generic way of refering to vectors since we only studied tensors of rank 1 (vectors) and rank 0 (scalars).

- Now we are going to study tensors of arbitrary rank k.