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Tensor Analysis (Ch. 4)

- A tensor of rank k in dimension N has N^k components.

- $k=0 \Rightarrow$ scalar 1 comp
- $k=1 \Rightarrow$ vector N comps.
- $k=2 \Rightarrow$ matrix N^2 comps

$\vdots$

Tensors and their components are defined by transformation rules going from S to S':

$k=0 \qquad \phi = \phi'$

For $k=1$ we found that

$$A^{i'} = \frac{\partial x'^{i}}{\partial x^{j}} A^{j}$$

contravariant components

sum over j

$$A_{i'} = \frac{\partial x^{j}}{\partial x'^{i}} A_{j}$$

covariant components.

sum over j

How do generate rank k tensors?

Direct product or outer product

$$A^{\mu} B^{\nu} = C^{\mu\nu}$$

C is a tensor of rank 2.

Example in 2D

$$A^{\mu} = (A^1, A^2) \qquad B^{\nu} = (B^1, B^2)$$

$$C^{\mu\nu} = \begin{pmatrix} A^1B^1 & A^1B^2 \\ A^2B^1 & A^2B^2 \end{pmatrix} = \begin{pmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{pmatrix}$$

How does $C^{\mu\nu}$ transform from S to S'?

$$C'^{\mu\nu} = A'^{\mu} B'^{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x'^{\nu}}{\partial x^{\beta}} B^{\beta} =$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} A^{\alpha} B^{\beta} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} C^{\alpha\beta}$$

$C^{\alpha\beta}$ is our prototype rank 2 contravariant tensor.

We can construct 4 kinds of rank 2 tensors by doing the outer product of two vectors:

$$C_{\mu\nu} = A_\mu B_\nu$$ covariant rank 2 tensor

and

$$C_\mu{}^\nu = A_\mu B^\nu$$
$$C^\mu{}_\nu = A^\mu B_\nu$$
mixed rank 2 tensors.

Transformation rules:

$$C'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} C_{\alpha\beta} \qquad \text{covariant}$$

$$C'^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} C^{\alpha}{}_{\beta} \qquad \text{mixed}$$

$$C'_{\mu}{}^{\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} C_{\alpha}{}^{\beta} \qquad \text{mixed}$$

Physical examples: electric quadrupole

$$Q^{ij} = \int_V \rho(\bar{x})\, x^i x^j \, d^3x$$

Higher rank tensors:

they can be constructed from the direct product of lower rank tensors.

$$C^{\mu\nu} B^{\sigma} = T^{\mu\nu\sigma} \qquad \text{rank 3}$$

The 3 indices transfor contravariantly:

$$T'^{\mu\nu\sigma} = C'^{\mu\nu} B'^{\sigma} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} C^{\alpha\beta} \frac{\partial x'^{\sigma}}{\partial x^{\gamma}} B^{\gamma} =$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \frac{\partial x'^{\sigma}}{\partial x^{\gamma}} T^{\alpha\beta\gamma}$$

Another possibility:

$$B^{\nu}{}_{\mu}{}^{\alpha}\, T_{\delta\varepsilon}{}^{\rho} = M^{\nu}{}_{\mu}{}^{\alpha}{}_{\delta\varepsilon}{}^{\rho}$$

mixed rank 6 tensor.

$$M'^{\nu}{}_{\mu}{}^{\alpha}{}_{\delta\varepsilon}{}^{\rho} = \frac{\partial x'^{\nu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial x'^{\mu}} \frac{\partial x'^{\alpha}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{\delta}} \frac{\partial x^{m}}{\partial x'^{\varepsilon}} \frac{\partial x'^{\rho}}{\partial x^{n}}$$

$$M^{i}{}_{j}{}^{k}{}_{lm}{}^{n}$$

In direct product the rank of the result is the sum of the ranks of the vectors that are being combined.

Addition of tensors:

We can add tensors of the same rank and with corresponding indices that transform in the same way:

$$C^{ab} = A^{ab} + B^{ab}$$

$$C^{a}{}_{b} = A^{a}{}_{b} + B^{a}{}_{b}$$

but

$$\cancel{A^{a}{}_{b} + B^{ab}}$$ wrong!

Contraction of tensors.

If two indices of a tensor of rank n (one index is covariant and the other contravariant) are set equal to each other implying a sum, the 2 indices are contracted and the result is another tensor of rank $n-2$.

Consider:

$$C^{\mu}{}_{\nu} = \begin{pmatrix} C^1{}_1 & C^1{}_2 \\ C^2{}_1 & C^2{}_2 \end{pmatrix} \qquad C^{\mu}{}_{\mu} = C^1{}_1 + C^2{}_2 \equiv \text{trace}(C^{\mu}{}_{\nu}) \quad \text{scalar}$$

Rank went from 2 to 0.

Warning!

- Tensors of rank 2 are matrices but not all matrices are tensors of rank 2.
- The trace of a tensor of rank 2 is a tensor only for $C^{\mu}{}_{\mu}$. $Tr(C^{\mu\mu})$ is <u>NOT</u> a tensor.

Example:

$$C^{\mu\nu} = x^{\mu} x^{\nu} \quad \text{and} \quad C^{\mu}{}_{\nu} = x^{\mu} x_{\nu}$$

Consider going from S: cartesian to $\bar{S}$: oblique.

$$C'^{\mu\nu} = \begin{pmatrix} x'^1 x'^1 & x'^1 x'^2 \\ x'^2 x'^1 & x'^2 x'^2 \end{pmatrix} \quad \text{and} \quad C'^{\mu}{}_{\nu} = \begin{pmatrix} x'^1 x'_1 & x'^1 x'_2 \\ x'^2 x'_1 & x'^2 x'_2 \end{pmatrix}$$

In S: $$C^{\mu}{}_{\nu} \doteq \begin{pmatrix} x_1^2 & x_1 x_2 \\ x_2 x_1 & x_2^2 \end{pmatrix}$$

$$C^{\mu}{}_{\mu} = x_1^2 + x_2^2 = r^2$$

In S': $C'^{\mu}{}_{\mu} = x'^1 x'_1 + x^{2\prime} x'_2 = r'^2 \equiv r^2$ invariant tensor of rank 0.

but $\text{trace}(C'^{\mu\nu}) = (x'^1)^2 + (x'^2)^2$ → scalar but not a tensor because it is not invariant.

We can see that $C^{\mu}{}_{\mu}$ is a tensor

because:

$$C'^{\mu}{}_{\mu} = A'^{\mu} B'_{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x^{\beta}}{\partial x'^{\mu}} B_{\beta} =$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} A^{\alpha} B_{\beta} =$$

$$= \underbrace{\frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\alpha}}}_{\text{chain rule}} A^{\alpha} B_{\beta} =$$

$$= \frac{\partial x^{\beta}}{\partial x^{\alpha}} A^{\alpha} B_{\beta} = \delta^{\beta}{}_{\alpha} A^{\alpha} B_{\beta} = A^{\alpha} B_{\alpha} =$$

$$= C^{\alpha}{}_{\alpha}$$

C is a scalar and a tensor of rank 0.

In general:

$$A^i{}_j \, B^{kl} = T^i{}_j{}^{kl}$$

I can contract

ij

jk

or

jl

I cannot contract

ik, il or kl

then

$$A^i{}_i \, B^{kl} = T^i{}_i{}^{kl} = M^{kl} \quad \text{rank } 2 = 4-2$$

Matrix multiplication and tensor contraction.

(in general they are not the same):

Matrix multiplication (not necessarily tensors)

you write:

① $C_{ij} = \sum_k A_{ik} B_{kj}$ $\mathbb{C} = \mathbb{A}\mathbb{B}$

If $\mathbb{C}$, $\mathbb{A}$ and $\mathbb{B}$ are tensors, what contraction corresponds to ①?

$C^i{}_j = A^{ik} B_{kj}$ using contravariant A and covariant B you can "multiply" and find mixed $\mathbb{C}$.

Also

$$C^{ij} = A^{i}{}_{k} B^{kj}$$

also like matrix multiplication

But

$$C^{ij} = A_{k}{}^{i} B^{kj}$$

C^{ij} is a tensor and looks like a matrix and $A_k{}^i$ and B^{kj} are tensors and matrices but C^{ij} does not result from matrix multiplication of A and B!

Tensor contractions vs similarity transformations:

A matrix in a basis S is transformed to a basis S' by doing:

$$A' = U A U^{-1}$$

where U is the change of basis matrix.

you write:

$$A'_{ij} = \sum_{k,l} U_{ik} A_{kl} (U^{-1})_{lj}$$

if you are going from system S to S' $U_{ij} = \frac{\partial x'^i}{\partial x^j}$

$$(U^{-1})_{\ell j} = \frac{\partial x^{\ell}}{\partial x'^{j}}$$

In terms of tensors (if A is a tensor):

$$(A')^{i}{}_{j} = \frac{\partial x'^{i}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{j}} A^{\alpha}{}_{\beta} = U^{i}{}_{\alpha} A^{\alpha}{}_{\beta} (U^{-1})^{\beta}{}_{j}$$

same as change of basis for A

but

$$A'^{ij} = \frac{\partial x'^{i}}{\partial x^{\alpha}} \frac{\partial x'^{j}}{\partial x^{\beta}} A^{\alpha\beta} = \underbrace{U^{i}{}_{\alpha} U^{j}{}_{\beta} A^{\alpha\beta}}_{\text{you can do it but NOT multiplying matrices.}}$$