Kronecker Delta

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Is it a tensor?

What kind?

We know that

$$\delta_{k\ell} = \begin{cases} 1 & \text{if } k=\ell \\ 0 & \text{if } k\neq\ell \end{cases}$$

In 2D:

$$\delta_{k\ell} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It has 2 indices so if it is a tensor it will have rank 2.

Let's see how $\delta_{k\ell}$ transforms from S to S':

In S':

chain rule since $x'^i = x'^i(x^j)$

$$\delta'_{ij} = \frac{\partial x'^i}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^k}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^\ell}{\partial x'^j}\,\delta_{k\ell}$$

contravariant for i

covariant for j

Then we obtained that

$\delta'_{ij} \equiv \delta'^i{}_j$ mixed rank 2 tensor.

Then $\delta'^i{}_j = \dfrac{\partial x'^i}{\partial x^k}\dfrac{\partial x^\ell}{\partial x'^j}\,\delta^k{}_\ell$.

Tensor properties:

In general $A^{nm} \neq A^{mn}$.

If A^{nm} and A^{mn} are components of a rank 2 tensor, in general, they are independent from each other.

However, some times $A^{nm} = A^{mn}$. In this case if it happens for all (n,m) then A^{nm} is symmetric.

B is a symmetric rank 2 tensor if

$$B^{nm} = B^{mn} \quad \forall\ m, n.$$

Antisymmetric tensor:

$$C^{nm} = -C^{mn}$$

Notice that $C^{aa} = -C^{aa} = 0. \quad \forall\ a.$

An anti-symmetric tensor is traceless.

Independent components

A tensor of rank 2 in dimension N has N^2 components. But not all of them may be independent.

- If B is symmetric.

$$n = \sum_{i=1}^{N} i = \frac{N(N+1)}{2}$$

$\hookrightarrow$ number of independent components.

Ex: $N=3$

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$N^2 = 9$

$$n = \frac{N(N+1)}{2} = \frac{3\times 4}{2} = 6$$

Antisymmetric tensor:

$$n = \frac{N(N+1)}{2} - N = \frac{N(N-1)}{2}$$

N elements a^{ii} are 0.

Example:

$$\begin{matrix} 0 & a_{12} & a_{13} \\ a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{matrix}$$

3 independent components

Every rank 2 tensor can be written as the sum of a symmetric and an antisymmetric tensor:

$$A = A_s + A_a$$

$$A^{mn} = \underbrace{\frac{1}{2}\left(A^{mn} + A^{nm}\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(A^{mn} - A^{nm}\right)}_{\text{antisymmetric}}$$

The symmetry properties of a tensor are intrinsic or independent of the system of coordinates.

Higher order tensors:

3D:

$$T^{ijkl} \qquad \text{rank } 4$$

$$N^4 = 3^4 = 81 \text{ components.}$$

But if $T^{\overset{6}{\widetilde{ij}}\overset{3}{k}\overset{3}{l}} = T^{jikl}$ symmetric for exchange of i and j.

Then

$$n = N \times N \times \frac{N(N+1)}{2} \overset{N=3}{=} 3 \times 3 \times 6 = 54$$

Example: Stress and elasticity tensors.

Hook's law: $\bar{F} = k\bar{x}$

In solids:

$$\underbrace{\sigma}_{\text{stress}} = \overbrace{C}^{\text{elasticity}} \underbrace{\varepsilon}_{\text{strain}}$$

In non-isotropic solids

$$\sigma_{\alpha\beta} = \lim_{\Delta A \to 0} \frac{\Delta F_\alpha}{\Delta A_\beta}$$

stress tensor

rank 2

Strain:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)$$

u_α : displacement from equilibrium.

elasticity tensor

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\,\varepsilon_{\gamma\delta}$$

$C_{\alpha\beta\gamma\delta}$: rank 4 tensor

It has $3^4 = 81$ components.

The 81 components are not independent.

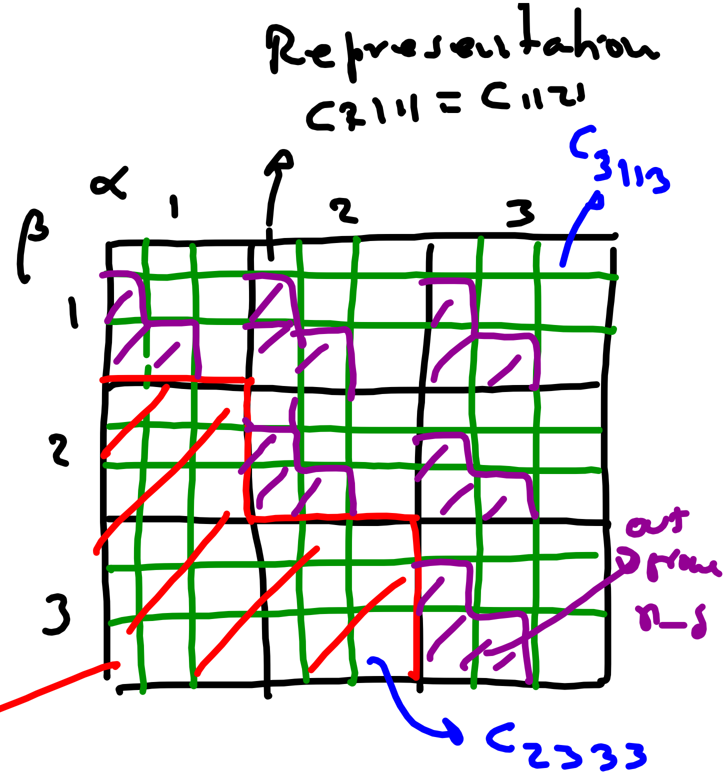
$\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$ and $\varepsilon_{\gamma\delta} = \varepsilon_{\delta\gamma}$ (both symmetric).

Then

$$C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\gamma\delta}$$

$$C_{\alpha\beta\gamma\delta} = C_{\alpha\beta\delta\gamma}$$

Only 36 independent elements left



I get these 3 blocks by exchanging $\alpha \leftrightarrow \beta$.

Also

$$C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta}$$

Each black square is a 3×3 matrix for $\gamma\delta$

$6+5+4+3+2+1=21$

$\boxed{C_{2111}}$ C_{2113} C_{2123}

C_{2112} C_{2122} C_{2133}

$C_{2111} = C_{1121} = C_{1112}$ (in first block)

Quotient Rule

We know that

$$\underbrace{A \ B}_{\text{direct product}} = C$$

for tensors with rank(C) = rank(A) + rank(B).

Now assume that we have

$$K A = B$$

with A, B : tensors

When is K a tensor?

If expression $KA = B$ is valid in S and $K'A' = B'$ is valid in S' then K is a tensor.

$$K_i A^i = B$$

$$K_i{}^j A_j = B_i$$

$$K^{ij} A^k = B^{ijk}$$

if A and B are tensors is K a tensor?

Assume that $K^{ij} A_j = B^i$ is also valid in S' so that $K'^{ij} A'_j = B'^i$ then we'll see that K transforms like a tensor. Here B is a known tensor and A is an arbitrary tensor.

$$K'^{ij} A'_j = B'^i = \frac{\partial x'^i}{\partial x^k} B^k = \frac{\partial x'^i}{\partial x^k}\left[K^{kl} A_l\right] =$$

because expression valid in S.

$$A'_j = \frac{\partial x^l}{\partial x'^j} A_l \qquad = \frac{\partial x'^i}{\partial x^k} K^{kl} \frac{\partial x'^j}{\partial x^l} A'^j =$$

$$A_l = \frac{\partial x'^j}{\partial x^l} A'_j \qquad = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} K^{kl} A'^j$$

$$\underbrace{\left(K'^{ij} - \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l} K^{kl}\right)}_{0} A'_j = 0$$

$A'_j \to$ arbitrary

Then

$$K'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l} K^{kl}$$

K^{ij} transforms like a rank 2 tensor.

Example:

In S you know that $\bar{p}$ (polarization) and $\bar{E}$ (electric field) are vectors and you find that

$$\bar{p} = P \bar{E}$$

as tensors $p^i = P^{ij} E_j$

If in S' you verify that

$\bar{p} = P' \bar{E}'$ then P^{ij} is a tensor of rank 2.