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Fundamental tensor: metric tensor.

g_{ij} is a tensor that will allow us to lower indices of a tensor. Its inverse g^{ij} ($g^{ij} \equiv (g_{ij})^{-1}$) will allow us to raise indices of a tensor.

g_{ij} can be any arbitrary symmetric rank 2 tensor that has an inverse ($\det g_{ij} \neq 0$).

But we are going to use the so called "metric tensor" as g_{ij}.

Metric Tensor!

- Let's consider a system S with coordinates $\{x^i\}$ and a system S' with coordinates $\{x'^j\}$.
- In S consider $d\bar{r} = dr^k$.
- In S' consider $d\bar{r}' = dr'^k$.

We know that $d\bar{r} = d\bar{r}'$ only their components are different.

- We can define vectors $\bar{\varepsilon}_j$ in S and $\bar{\varepsilon}'_j$ in S' so that they are the covariant basis vectors for each system.

- Then

$$d\bar{r} = \bar{\varepsilon}_j \, dx^j \quad (1)$$

and

$$d\bar{r}' = \bar{\varepsilon}'_j \, dx'^j \quad (2)$$

In cartesian coordinates

$$\bar{\varepsilon}_i \equiv \hat{e}_i \text{ (unit vector)}$$

- Then:

From (1): $\bar{\varepsilon}_j = \frac{d\bar{r}}{dx^j}$

From (2): $\bar{\varepsilon}_j = \frac{d\bar{r}'}{dx'^j}$

Notice that the "components" of $\bar{\varepsilon}_j$ are:

$$(\bar{\varepsilon}_j)^k$$

In tensor notation:

$$\bar{r} = r^k \qquad \text{then} \qquad \bar{\varepsilon}_j = (\bar{\varepsilon}_j)^k$$

- Now let's define the rank 2 tensor g_{ij} as the direct product of the basis vectors:

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j \equiv (\varepsilon_i)^k (\varepsilon_j)_k$$

scalar
but component of a tensor of rank 2.

In cartesian coordinates:

$$\bar{r} = x^k \hat{e}_k = x^1 \hat{e}_1 + x^2 \hat{e}_2 + x^3 \hat{e}_3$$

tensor notation $\bar{r} = r^k \equiv x^k = (x^1, x^2, x^3)$

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j = \begin{pmatrix} \hat{e}_1 \cdot \hat{e}_1 & \hat{e}_1 \cdot \hat{e}_2 & \hat{e}_1 \cdot \hat{e}_3 \\ \hat{e}_2 \cdot \hat{e}_1 & \hat{e}_2 \cdot \hat{e}_2 & \hat{e}_2 \cdot \hat{e}_3 \\ \hat{e}_3 \cdot \hat{e}_1 & \hat{e}_3 \cdot \hat{e}_2 & \hat{e}_3 \cdot \hat{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \mathbb{I}$$

In cartesian coordinates g_{ij} is the identity.

Let's see that $g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j$ is a tensor:

In S:

$$ds^2 = d\bar{r} \cdot d\bar{r} = \frac{\partial \bar{r}}{\partial x^i} \cdot \frac{\partial \bar{r}}{\partial x^j} \, dx^i dx^j \overset{①}{=} \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j dx^i dx^j$$

$$= g_{ij} \underbrace{dx^i dx^j}_{\text{vectors: tensors of rank 1}}$$

scalar rank 0

We know that

$$d\bar{r} = \frac{\partial \bar{r}}{\partial x^i} dx^i$$

also

$$d\bar{r} = \bar{\varepsilon}_j dx^j$$

then $\bar{\varepsilon}_j = \frac{\partial \bar{r}}{\partial x^j}$ ①

g_{ij} has to be a rank 2 tensor using the quotient rule.

- Inverse of the metric tensor:

Let's define:

g^{ij} so that $g^{ij} g_{jk} = \delta^i{}_k$

Then we see that

$g^{ij} = (g_{jk})^{-1}$.

Now define that

$$g^{ik} A_k = A^i \qquad \text{(definition).}$$

then:

$$g_{ik} A^k = g_{ik} \left[g^{kl} A_l \right] = \underbrace{g_{ik} g^{kl}}_{\delta_i^{\,l}} A_l = A_i$$

We see that g^{ik} raises indices and g_{ik} lowers indices.

What happens in S'?

If in S $\quad g^{ij} g_{jk} = \delta^i{}_k$

In S':

$$g'^{ij} g'_{jk} = \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta} g^{\alpha\beta} \frac{\partial x^\gamma}{\partial x'^j} \frac{\partial x^\delta}{\partial x'^k} g_{\gamma\delta} =$$

$$= \frac{\partial x'^i}{\partial x^\alpha} \underbrace{\frac{\partial x^\gamma}{\partial x'^j} \frac{\partial x'^j}{\partial x^\beta}}_{\frac{\partial x^\gamma}{\partial x^\beta} = \delta^\gamma{}_\beta} \frac{\partial x^\delta}{\partial x'^k} g^{\alpha\beta} g_{\gamma\delta} =$$

$$= \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x^\delta}{\partial x'^k} \overbrace{g^{\alpha\beta} g_{\beta\delta}}^{\delta^\alpha{}_\delta} = \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^k} = \frac{\partial x'^i}{\partial x'^k} = \delta'^i{}_k$$

Then we saw that in S':

$$g'^{ij} g'_{jk} = \delta'^{i}{}_{k}$$ because g^{ij} and g_{ij} are tensors.

- Properties of g_{ij}:

  - $g_{ij} = g_{ji} \Rightarrow g_{ij}$ is symmetric.

D/ $g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j = \bar{\varepsilon}_j \cdot \bar{\varepsilon}_i = g_{ji}$

then $g_{ij} A^j = g_{ji} A^j = A_i$

Examples:

1) Metric tensor in polar coordinates.

- Consider system S with coordinates $(x^1, x^2) \equiv (\rho, \varphi)$.
- and S' with coordinates $(x'^1, x'^2) \equiv (x, y)$
- Transformations:

$x^i(x'^j)$:

$$\begin{cases} \rho = (x^2 + y^2)^{1/2} \\ \varphi = \tan^{-1}(y/x) \end{cases}$$

$x'^j(x^i)$:

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \end{cases}$$

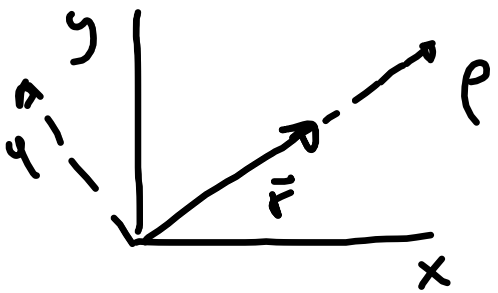
Notice that

$$\bar{r}' = (x, y) = (\rho \cos\varphi, \rho \sin\varphi) = \rho \cos\varphi\, \hat{e}_1 + \rho \sin\varphi\, \hat{e}_2 =$$

$$= \rho\, \bar{\varepsilon}_\rho + \varphi\, \bar{\varepsilon}_\varphi = \bar{r}$$

Then • $\bar{\varepsilon}_\rho = \dfrac{\partial \bar{r}}{\partial \rho} = (\cos\varphi, \sin\varphi)$ Cartesian components of $\bar{\varepsilon}_\rho$

$$|\bar{\varepsilon}_\rho| = 1$$



• $\bar{\varepsilon}_\varphi = \dfrac{\partial \bar{r}}{\partial \varphi} = (-\rho \sin\varphi, \rho \cos\varphi) =$

$$= \rho(-\sin\varphi, \cos\varphi)$$

$$|\bar{\varepsilon}_\varphi| = \rho \neq 1!$$

Now

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j = \begin{pmatrix} \bar{\varepsilon}_\rho \cdot \bar{\varepsilon}_\rho & \bar{\varepsilon}_\rho \cdot \bar{\varepsilon}_\varphi \\ \bar{\varepsilon}_\varphi \cdot \bar{\varepsilon}_\rho & \bar{\varepsilon}_\varphi \cdot \bar{\varepsilon}_\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}$$

g_{ij} is diagonal because S is an orthogonal system of coordinates but $g_{ij} \neq \mathbb{I}$.

then

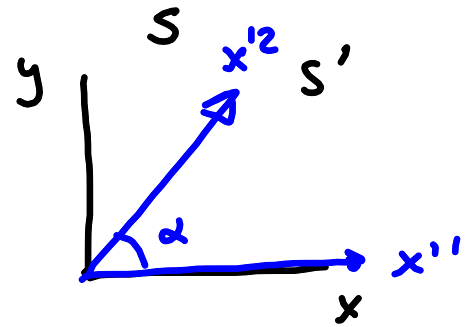
$$g^{ij} = (g_{ij})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\rho^2 \end{pmatrix} \neq g_{ij}$$

2) Oblique system:

Now S : cartesian system $(x^1, x^2) = (x, y)$

S': oblique system (x'^1, x'^2)

In S: $g_{ij} = g^{ij} = \mathbb{I}$



In S': $g'_{ij} = \bar{\varepsilon}'_i \cdot \bar{\varepsilon}'_j$

In homework you found that

$\bar{\varepsilon}'_1 = \hat{e}_1 = (1, 0)$ $\bar{\varepsilon}'_2 = (\cos\alpha, \sin\alpha)$

then

$$g'_{ij} = \begin{pmatrix} \bar{\varepsilon}'_1 \cdot \bar{\varepsilon}'_1 & \bar{\varepsilon}'_1 \cdot \bar{\varepsilon}'_2 \\ \bar{\varepsilon}'_2 \cdot \bar{\varepsilon}'_1 & \bar{\varepsilon}'_2 \cdot \bar{\varepsilon}'_2 \end{pmatrix} = \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}$$ symmetric

non diagonal because S' it is not orthogonal

if $\alpha = \frac{\pi}{2} \Rightarrow S'$ orthogonal then g'_{ij} diag.

then

$$g'^{ij} = (g'_{ij})^{-1} = \frac{1}{\sin^2\alpha} \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}$$

Notice that $x'_i = g_{ij}\, x'^j$!

$x'_1 = g'_{11} x'^1 + g'_{12} x'^2 =$
$= x'^1 + \cos\alpha\, x'^2 = x'_1$
same for x'_2

Then we see that for scalar product we can write:

$$a^{\mu} b_{\mu} = g^{\mu\nu} a_{\nu} b_{\mu} = g_{\mu\nu} a^{\mu} b^{\nu}$$

If $g^{\mu\nu} = \mathbb{I}$ it is irrelevant where you put the indices but if $g_{\mu\nu} \neq \mathbb{I}$ it really matters!!