9/26

Levi-Civita Tensor

In 3D

$$\varepsilon^{ijk} = \varepsilon_{ijk} = \begin{cases} 1 & \text{if } i \neq j \neq k \rightarrow \text{cyclic order } 123, 231, \text{etc.} \\ -1 & \text{if } i \neq j \neq k \rightarrow \text{not in cyclic order} \\ 0 & \text{otherwise} \end{cases}$$

cartesian

- Rank 3
- Totally antisymmetric, i.e. antisymmetric under exchange of any pair of indices.
- $3^3 = 27$ components.
- 6 non-zero components.
- 1 independent component From $\varepsilon_{123} = 1$ you get all ε_{ijk}.

In dimension N Levi-Civita is a tensor of rank N.

$$\varepsilon_{i_1, i_2, \ldots, i_N}$$

$$\_\_\_\_\_\_ \times \_\_\_\_\_\_$$

Properties in 3D:

$$\varepsilon_{pqr} = \hat{x}^p \cdot (\hat{x}^q \times \hat{x}^r)$$

$\hat{x}^a$: coordinate unit vectors.

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & & \vdots \\ a_{31} & \cdots & a_{33} \end{vmatrix} = a_{1i} a_{2j} a_{3k} \varepsilon_{ijk}$$

only six terms in the sum.

Expand det A and compare.

- In dimension N

$$\det A = \begin{vmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \\ a_{N1} & \cdots & a_{NN} \end{vmatrix} = a_{1i} a_{2j} \cdots a_{Nz} \, \varepsilon_{ij \ldots z}$$

- Also

$$\det A \, \varepsilon_{\alpha\beta\gamma} = a_{\alpha i} a_{\beta j} a_{\gamma k} \varepsilon_{ijk} \quad (*)$$

$$\parallel$$

$$a_{1i} a_{2j} a_{3k} \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma}$$

Shown in homework by expanding.

- Also $C_i = \varepsilon_{ijk} A^j B^k \equiv \bar{C} = \bar{A} \times \bar{B}$

Tensors and Pseudotensors.

We have study how tensors transform going from a system S to a system S'. We have been considering our prototype contravariant and covariant vectors r^i and $\partial_i \phi$. However there are some tensors that transform in a special way if $\det|A|$ (with $A^i{}_j = \frac{\partial x'^i}{\partial x^j}$) is equal to -1.

Inversion:

$S : \{ x^i \}$ $\qquad S' : \{ x'^i \}$ $\qquad x'^i = -x^i$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad A^i{}_j = \frac{\partial x'^i}{\partial x^j}$$

$\det A = -1$

Consider r^i:

$$\bar{r} = (x^1, x^2, x^3) \quad \bar{r}' = (x^{1'}, x^{2'}, x^{3'}) = (-x^1, -x^2, -x^3)$$

$$r'^i = \frac{\partial x'^i}{\partial x^j} r^j$$

$\bar{r}$ is called a polar vector - Its coordinates change sign upon an inversion of the system of coordinates.

Now consider:

$$\bar{C} = \bar{A} \times \bar{B} \qquad \bar{A}, \bar{B} \text{ polar vectors}$$

In components:

$$C^i = A^j B^k - A^k B^j \quad \text{in } S$$

If we go to S' (inverted system)

$A'^j = -A^j \quad B'^k = -B^k$ but $C'^i = C^i$

We see that

$$A'^j = \frac{\partial x'^j}{\partial x^i} A^i = -\delta^j{}_i A^i = -A^j$$

But

$$C'^j = (\det A) \frac{\partial x'^j}{\partial x^i} C^i = -(-1)\,\delta^j{}_i C^i = +\; C^i$$

if A has det A = 1 as if S' is a rotated system then C^i transforms like a polar vector r^i.

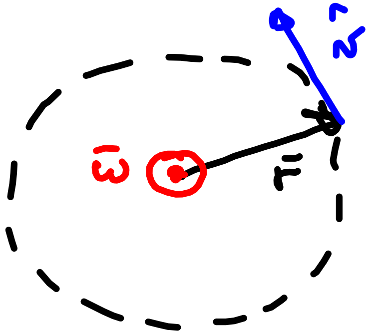
But if det A = −1 as in inversion C^i transforms differently.

Vectors that transform like C^i are called axial or pseudovectors.

Examples in physics:

$$\bar{v} = \bar{\omega} \times \bar{r}$$

$$\bar{\omega} = \frac{\bar{r} \times \bar{v}}{|r|^2}$$



$$\bar{r} \times \bar{v} = \bar{r} \times (\bar{\omega} \times \bar{r}) =$$

$$= \underbrace{\bar{\omega}(\bar{r}\cdot\bar{r})}_{\bar{\omega}\, r^2} - \underbrace{\bar{r}(\bar{r}\cdot\bar{\omega})}_{0}$$

$\bar{r}, \bar{v}$ are polar vectors.

You see that if $\bar{r} \to -\bar{r}$ and $\bar{v}_r \to -\bar{v}$

$\bar{\omega} \to \bar{\omega}$. So $\bar{\omega}$ is an axial vector.

$\bar{L} = \bar{r} \times \bar{p}$ is axial vector since $\bar{r}$ and $\bar{p}$ are polar.

$L_i = \varepsilon_{ijk} r^j p^k$

How does ε_{ijk} transform?

$$\varepsilon'^{ijk} = \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta} \frac{\partial x'^k}{\partial x^\gamma} \varepsilon^{\alpha\beta\gamma} =$$

$$= A^i{}_\alpha A^j{}_\beta A^k{}_\gamma \varepsilon^{\alpha\beta\gamma} = \quad (*)$$

$$= \det A \ \varepsilon^{ijk}$$

if $\det A = 1$ then $\varepsilon'^{ijk} = \varepsilon^{ijk}$ isotropic tensor

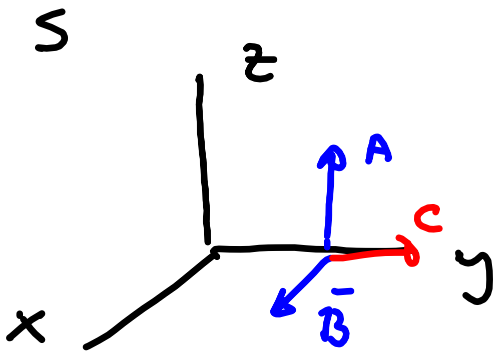
To ensure that ε^{ijk} remains isotropic if $\det A = -1$ we define:

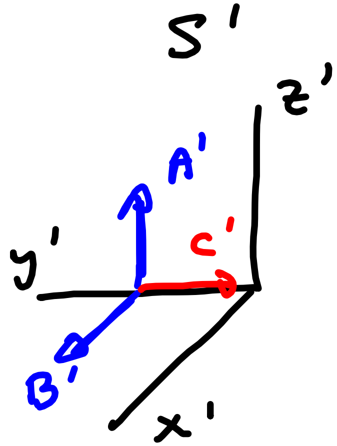
$$\varepsilon'^{ijk} = \det A \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta} \frac{\partial x'^k}{\partial x^\gamma} \varepsilon^{\alpha\beta\gamma}$$

ε^{ijk} transforms like a pseudotensor.

Then it is a pseudotensor.

Reflections.





C' point along $-y'$ so its direction is reversed.

$$x^{1'} = x^1 \qquad x^{2'} = -x^2$$

$$x^{3'} = x^3$$

$$A = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix} \Rightarrow \det A = -1$$

$$r'^i = \frac{\partial x'^i}{\partial x^j} r^j$$

or

$$x'^i = \frac{\partial x'^i}{\partial x^j} x^j$$

Properties of Cross-products:

$$\bar{C} = \bar{A} \times \bar{B}$$

$$C^{i} = \varepsilon^{ijk} A_j B_k$$

$\bar{A}$, $\bar{B}$ are vectors

their components change sign upon inversion.

Then $\bar{C}$ will be a pseudovector.

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$$C'^{i} = \varepsilon'^{ijk} A'_j B'_k = \det A \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta} \frac{\partial x'^k}{\partial x^\gamma} \varepsilon^{\alpha\beta\gamma}$$

$$\frac{\partial x^\delta}{\partial x'^j} A_\delta \frac{\partial x^\varepsilon}{\partial x'^k} B_\varepsilon =$$

$$= \det A \frac{\partial x'^i}{\partial x^\alpha} \underbrace{\frac{\partial x^\delta}{\partial x'^j} \frac{\partial x'^j}{\partial x^\beta}}_{\frac{\partial x^\delta}{\partial x^\beta} = \delta^\delta{}_\beta} \underbrace{\frac{\partial x^\varepsilon}{\partial x'^k} \frac{\partial x'^k}{\partial x^\gamma}}_{\frac{\partial x^\varepsilon}{\partial x^\gamma} = \delta^\varepsilon{}_\gamma} \varepsilon^{\alpha\beta\gamma} A_\delta B_\varepsilon =$$

$$= \det A \frac{\partial x'^i}{\partial x^\alpha} \varepsilon^{\alpha\beta\gamma} A_\beta B_\gamma \quad D$$

Then

$$C'^{i} = \underline{\underline{\underline{\det A}}} \frac{\partial x'^{i}}{\partial x^{\alpha}} C^{\alpha}$$

pseudovector

Notice that

$$A \times B = C$$

$$p \times p = a$$

$$p \times a = p$$

$$a \times p = p$$

$$a \times a = a$$

Summarizing:

scalars

$S' = S$ in all systems

pseudoscalars

$S' = \det A \; S$

may change sign if $\det A = -1$.

vector

$$c'_i = \frac{\partial x^j}{\partial x'^i} c_j$$

Pseudovector:

$$c'_i = \det A \frac{\partial x^j}{\partial x'^i} c_j$$

tensor

$$T'^{ijk\cdots} = a^i{}_l \, a^j{}_m \, a^k{}_n \cdots T^{lmn\cdots}$$

pseudo tensor

$$T'^{ijk\cdots} = \det A \cdots\cdots$$

$T \otimes T = T$ $\qquad$ $T \otimes P = P$

$P \otimes P = T$ $\qquad$ $P \otimes T = P$