Last time: 9/5

$$r'^i = \frac{\partial x'^i}{\partial x^j} r^j$$ contravariant vector

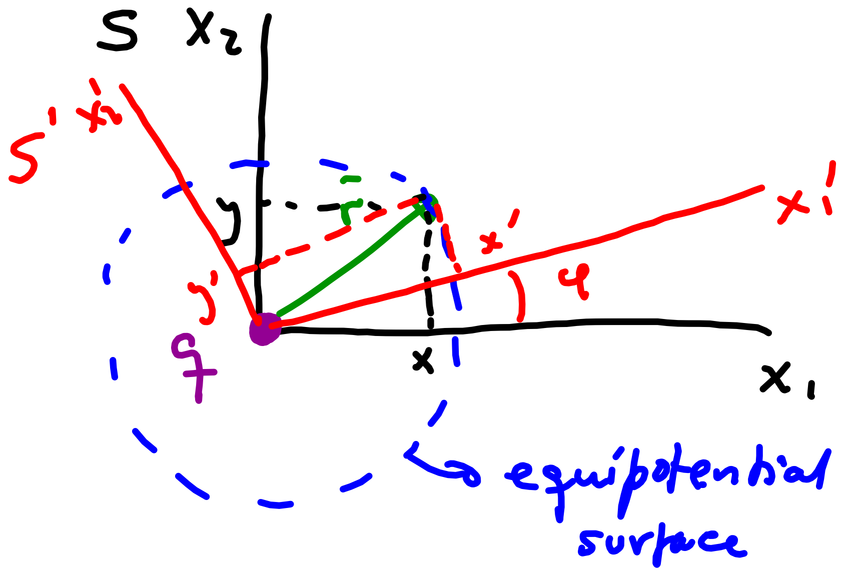
$$\partial'_i F'(x'^k) = B'_i = \frac{\partial x^j}{\partial x'_i} B_j$$ covariant vector

$$= \frac{\partial x^j}{\partial x'_i} \partial_j F(x^k)$$

$$\partial_i \equiv \frac{\partial}{\partial x'^i}$$ covariant derivative is a derivative with respect to the contravariant components.

Example of covariant vector in 2D. The electric field $\bar{E}(\bar{x})$.

Consider:



In S: q produces a potential

$$\phi_q = \frac{A}{r} = \phi_q(\bar{r}) = \frac{A}{(x^2+y^2)^{1/2}} = V$$

In S':

$$\phi_q'(x',y') = \frac{A}{(x'^2+y'^2)^{1/2}} = V$$

Since ϕ_q is an scalar $\phi_q = \phi_q'$

Now let's calculate $\bar{E}$ in S:

$$\bar{E}(x,y) = -\bar{\nabla}\phi(x,y)$$

$$-\frac{\partial}{\partial x_i}\left(\frac{A}{(x^2+y^2)^{1/2}}\right)$$

In S'

$$\bar{E}'(x',y') = -\bar{\nabla}'\phi'(x',y')$$

$$\bar{E}(\bar{r}) = (E_x, E_y) = \left(-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}\right) = \frac{A}{r^3}(x,y)$$

$$\bar{E}'(\bar{r}') = (E'_{x'}, E'_{y'}) = \left(-\frac{\partial\phi'}{\partial x'}, -\frac{\partial\phi'}{\partial y'}\right) = \frac{A}{r'^3}(x',y')$$

Let's find the relationship between $\bar{E}$ and $\bar{E}'$:

$$E'_i(\bar{r}') = -\frac{\partial \phi'(x',y')}{\partial x^{i'}} = -\frac{\partial \phi(x,y)}{\partial x^{i'}} =$$

$$= -\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial x'^i} - \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial x'^i} =$$

$$= \underbrace{-\frac{\partial \phi}{\partial x^j}}_{E_j(x,y)}\frac{\partial x^j}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} E_j$$

transforms as a covector

$[\phi] = \text{Volts}$

$[\bar{E}] = \text{Volts/m}$ (1/length means covector)

Another example: Now S' is the oblique system.

In S':

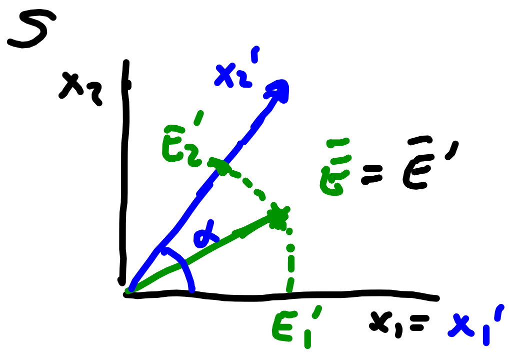
$$E'_i = E_j \frac{\partial x^j}{\partial x'^i}$$

$$\therefore \quad (E'_1, E'_2) = (E_1, E_2) \underbrace{\begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix}}_{A^j{}_i}$$

$$\Rightarrow E'_1 = E_1$$

$$E'_2 = E_1 \cos\alpha + E_2 \sin\alpha$$

perpendicular projections.



Let's look at the expression for the work done by the electric force:

covariant

$$dW = \bar{F} \cdot \bar{dl} = q\,\bar{E} \cdot d\bar{r} = q\,E_i\,dr^i$$

contravariant

as matrices:

$$q\,(E_1, E_2)\begin{pmatrix} dx \\ dy \end{pmatrix} = dW$$

Notation:

$$\frac{\partial}{\partial x^i} \equiv \partial_i$$ covector (derivative with respect to the contravariant components)

Gradient:

$$\bar{\nabla} F = \bar{B} \equiv B_i = \partial_i F$$ covector

Total differential:

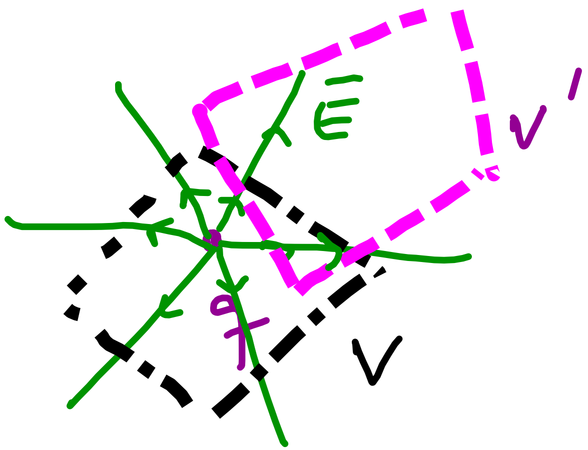
$$dF = \bar{\nabla} F \cdot d\bar{r} = \partial_i F \, dr^i$$ scalar

Divergence:

$$\bar{\nabla} \cdot \bar{G} = \partial_i G^i \equiv \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

Refresher:

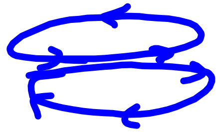
Divergence: A field $\bar{G}$ has non-zero divergence in a volume V if it has a singularity inside V.



In V $\bar{\nabla}\cdot\bar{E} \neq 0$.

In V' $\bar{\nabla}\cdot\bar{E} = 0$.

Fields with $\bar{\nabla}\cdot\bar{G} = 0$ everywhere are called solenoidal. Example:



$\bar{\nabla}\cdot\bar{B} = 0$ no magnetic monopoles.

Use of tensor notation to obtain useful expressions:

1) $\bar{\nabla} \cdot \bar{r} = \partial_i x^i = N$ (3 in 3 dimensions)

2) $\bar{\nabla} \cdot [\bar{r} f(r)] = \partial_i [x^i f(r)] = f(r) \underbrace{\partial_i x^i}_{3} +$ (N=3)

scalar function of $|\bar{r}|$

$x^i x_i = r^2$

$$+ x^i \underbrace{\partial_i f(r)}_{\frac{\partial f}{\partial r} \frac{\partial r}{\partial x^i}} = 3 f(r) + x^i \frac{\partial f}{\partial r} \underbrace{\partial_i r}_{x_i / r} = 3 f(r) + r \frac{\partial f}{\partial r}$$

$$\partial_i r = \partial_i (x_j x^j)^{1/2} = \frac{1}{2} \frac{1}{\underbrace{(x_j x^j)^{1/2}}_{r}} \Big[\overbrace{\underbrace{\partial_i x_j \, x^j}_{x_j \delta_i{}^j}}^{\text{in cartesians } x_i = x^i \text{ then}} + x_j \overbrace{\partial_i x^j}^{\delta_i{}^j} \Big] = \frac{x_i}{r}$$

3) Application of (2):

$$\bar{\nabla} \cdot (\bar{r}\, \underbrace{r^{n-1}}_{f(r)}) = 3\, r^{n-1} + r\, \frac{\partial (r^{n-1})}{\partial r} =$$

$$= 3\, r^{n-1} + r\, (n-1)\, r^{n-2} = (n+2)\, r^{n-1}$$

More review!

Curl

$$\bar{\nabla} \times \bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \hat{i} +$$

$$+ \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \hat{k}$$

Examples!

$\bar{B} = \bar{\nabla} \times \bar{A}$ ($\bar{A}$: vector potential)

$\bar{\nabla} \times \bar{E} = 0$ (for electrostatic field)

Example:

$$\bar\nabla \times (\bar r\, f(r)) = \ ? \qquad ①$$

$f(r) \hookrightarrow$ central force

In the back of the book:

$$\bar\nabla \times (f \bar V) = f\, \bar\nabla \times \bar V + (\bar\nabla f) \times \bar V$$

$f \to$ scalar, $\bar V \to$ vector

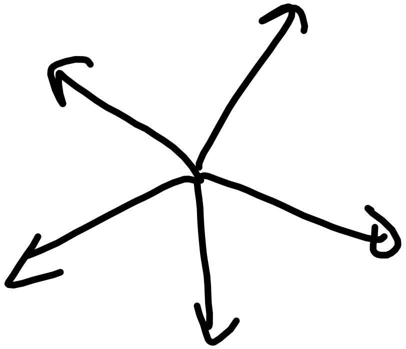
To solve ①:

$$\bar\nabla \times (\bar r f(r)) = f(\bar r)\, \overbrace{\bar\nabla \times \bar r}^{0} + \overbrace{(\underbrace{\bar\nabla f(r)}) \times \bar r}^{0} = 0$$

$$\bar\nabla \times \bar r = \begin{vmatrix} \hat i & \hat j & \hat k \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = 0$$

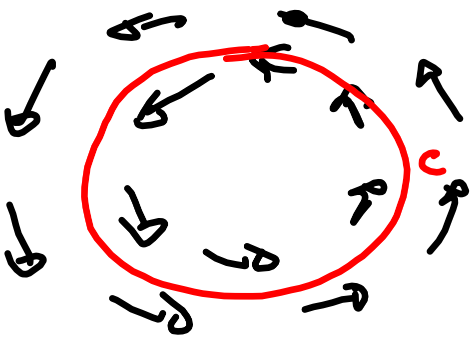
$f(r)$ constant for $x^2 + y^2 \ldots$ ∴ $\bar\nabla f(r) \parallel \bar r$

A field $\bar{G}$ with $\bar{\nabla} \times \bar{G} = 0$ is called irrotational



irrotational

If $\bar{\nabla} \times \bar{G} \neq 0$ $\bar{G}$ is rotational.



$\oint_C \bar{G} \cdot \bar{dl} \neq 0$ It has circulation

Laplacian:

$$\nabla^2 = \bar{\nabla} \cdot \bar{\nabla}$$ (scalar)

$$\nabla^2 \varphi = \overbrace{\bar{\nabla} \cdot (\underbrace{\bar{\nabla} \varphi}_{\text{gradient}})}^{\text{divergence}} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

scalar

In tensor form it is:

$$\nabla^2 \equiv \partial_i \partial^i$$

(we'll learn later how to obtain ∂^i)

in cartesians $\partial_i = \partial^i$