Midterm Exam

P571
October 3, 2013

SOLUTION:

a) We need to express the coordinates of **r'** in S' in terms of the coordinates of **r** in S. We see that the z component of **r** is not affected by the change of system. Thus we need to concentrate on the components of **r** parallel to the plane perpendicular to $z = x_3$ as shown in the figure:

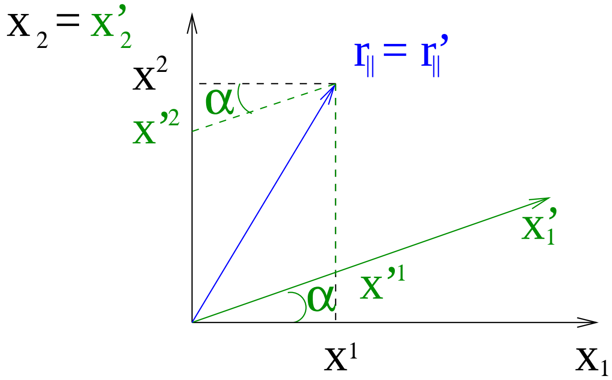

FIG. 1: Decomposition of the parralel component of **r** in S and S'.

We see that

$$x'^1 \cos\alpha = x^1, \tag{1}$$

$$x^2 - x'^2 = x'^1 \sin\alpha. \tag{2}$$

Then, solving for x'^1 in Eq.(1) we obtain:

$$x'^1 = \sec\alpha x^1, \tag{3}$$

and plugging Eq.(3) in Eq.(2) and solving for x'^2 we obtain:

$$x'^2 = x^2 - \tan\alpha x^1, \tag{4}$$

and

$$x'^3 = x^3. \tag{5}$$

b) We can obtain $M^i{}_j = \frac{\partial x'^i}{\partial x^j}$ from Eqs. (3,4,5):

$$M^i{}_j = \begin{pmatrix} \sec\alpha & 0 & 0 \\ -\tan\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

c) The covariant basis in S is given by:

$$\hat{\mathbf{e}}_1 = (1, 0, 0), \tag{7}$$

$$\hat{\mathbf{e}}_2 = (0,1,0), \tag{8}$$

$$\hat{\mathbf{e}}_3 = (0,0,1). \tag{9}$$

d) The easiest way is to use the transformation rules:

$$\hat{\mathbf{e}}_i = \frac{\partial x'^j}{\partial x^i}\hat{\mathbf{e}'}_j. \tag{10}$$

We need to solve the system of equations given by:

$$\hat{\mathbf{e}}_1 = \sec\alpha\hat{\mathbf{e}'}_1 - \tan\alpha\hat{\mathbf{e}'}_2, \tag{11}$$

$$\hat{\mathbf{e}}_2 = \hat{\mathbf{e}'}_2, \tag{12}$$

and

$$\hat{\mathbf{e}}_3 = \hat{\mathbf{e}'}_3. \tag{12}$$

We obtain:

$$\hat{\mathbf{e}'}_1 = \cos\alpha\hat{\mathbf{e}}_1 + \sin\alpha\hat{\mathbf{e}}_2, , \tag{13}$$

$$\hat{\mathbf{e}'}_2 = \hat{\mathbf{e}}_2, \tag{14}$$

and

$$\hat{\mathbf{e}'}_3 = \hat{\mathbf{e}}_3. \tag{15}$$

e) We know that

$$g'_{ij} = \hat{\mathbf{e}}_i.\hat{\mathbf{e}}_j = \begin{pmatrix} 1 & \sin\alpha & 0 \\ \sin\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

f) We know that

$$x'_i = g'_{ij}x'^j, \tag{17}$$

and in part (a) we obtained x'^j in terms of x^i then we obtain Then,

$$x'_1 = x^1\cos\alpha + x^2\sin\alpha, \tag{18}$$

$$x'_2 = x^2, \tag{19}$$

and

$$x'_3 = x^3. \tag{20}$$

g) We know that in S $\mathbf{r}$=$(x^1,x^2,x^3)=(3,2,1)$ then from (a) we find that

$$x'^1 = 3\sqrt{2}, \tag{21}$$

$$x'^2 = -1, \tag{22}$$

and

$$x'^3 = 1. \tag{23}$$

The relationship between x^i and x'_j was obtained in (f) then using these results we obtain:

$$x'_1 = 5\frac{\sqrt{2}}{2}, \tag{24}$$

$$x'_2 = 2, \tag{25}$$

and

$$x'_3 = 1. \tag{26}$$

h) The results obtained in (g) allow to calculate

$$x'_i x'^i = 15 - 2 + 1 = 14. \tag{27}$$

In S we obtain

$$x_i x^i = 9 + 4 + 1 = 14. \tag{28}$$

As expected the results are equal in both systems,

i) The tensor $T'^{ij}{}_{kl}$ has rank 4 because it has 4 indices. Since we are in three dimensions the tensor has $3^4 = 81$ components.

j) We need to use the transformation rules to go from S' to S for each index then:

$$T^{mn}{}_{op} = \frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j}\frac{\partial x'^k}{\partial x^o}\frac{\partial x'^l}{\partial x^p}T'^{ij}{}_{kl}. \tag{29}$$

k) I know that

$$T'^{12}{}_{21} = x'^1 x'^2 x'_2 x'_1 = 3\sqrt{2} \times (-1) \times 2 \times 5\sqrt{2}/2 = -30, \tag{30}$$

while

$$T^{12}{}_{21} = x^1 x^2 x_2 x_1 = 3 \times 2 \times 2 \times 3 = 36. \tag{31}$$

Since they are the components of a tensor we do not expect the two to be equal.

l) We see that T^{mnop} is a tensor symmetric under the exchange of any pair of indices. This means that of its 3x3x3x3=81 components only 15 of them are independent. We see that the independent components can be counted in the following way:

i) Components with all the indices equal, i.e., iiii: 3
ii) Components with 3 equal indeces and one different, iiij: 6 independent (3 values for i and 2 values left for j).
iii) Components with indices equal in pairs, iijj: 3 independent.
iv) Components with 2 equal indices and the other 2 different, iijk: 3 independent.

m) The tensor ${T'^{ij}}_{ij}$ has rank 0 because all its indices are contracted, i.e., it is a scalar.

n) We know that

$$ {T'^{ij}}_{ij} = r'^i r'^j r'_i r'_j = x'^i x'_i x'^j x'_j = 14 \times 14 = 196, \tag{32} $$

where I used the result of part (h) Eq.(27).

o) We know that

$$ {T^{ij}}_{ij} = r^i r^j r_i r_j = x^i x_i x^j x_j = 14 \times 14 = 196, \tag{33} $$

where I used the result of part (h) Eq.(28). we see that $T = T'$ as it should be since the tensor is a scalar.