Midterm Exam II

P571
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SOLUTION:

Problem 1:

a) We need to write in tensor form the LHS and the RHS of $[(\mathbf{A}\times\mathbf{B})\times\mathbf{B}].\mathbf{A} = (\mathbf{A}.\mathbf{B})^2 - A^2B^2$ and show that they are equivalent.

The RHS in tensor form is given by:

$$A^iB_iA^jB_j - A_tA^tB_uB^u, \tag{1}$$

while the LHS in tensor form is given by:

$$\epsilon^{klm}\epsilon_{lrs}A^rB^sB_mA_k = \epsilon^{lmk}\epsilon_{lrs}A^rB^sB_mA_k = (\delta^m{}_r\delta^k{}_s - \delta^m{}_s\delta^k{}_r)A^rB^sB_mA_k = (A^mB^kB_mA_k - A^kB^mB_mA_k) =$$

$$A^mB_mB^kA_k - A^kA_kB^mB_m. \tag{2}$$

Since m and k are dummy indices in the last term of Eq.(2) we see that the result matches Eq.(1) proving the equality.

b) From Eq.(2) we see that the expression does not have any free indices and, thus, it is a tensor of rank 0.

c) Now we need to consider how the tensor $T = \epsilon^{klm}\epsilon_{lrs}A^rB^sB_mA_k$ of rank 0, transforms from a system S to a system S'. If $\mathbf{A}$ is a polar vector it transforms as

$$A'^r = \frac{\partial x'^r}{\partial x^j}A^j, \tag{3}$$

while

$$A'_k = \frac{\partial x^q}{\partial x'^k}A_q. \tag{4}$$

The axial vector $\mathbf{B}$ transforms as

$$B'^s = |detM|\frac{\partial x'^s}{\partial x^y}B^y, \tag{5}$$

while

$$B'_m = |detM|\frac{\partial x^w}{\partial x'^m}B_w. \tag{6}$$

ϵ^{klm} and ϵ_{lrs} are axial tensors that transform like

$$\epsilon'^{klm} = |detM|\frac{\partial x'^k}{\partial x^n}\frac{\partial x'^l}{\partial x^o}\frac{\partial x'^m}{\partial x^p}\epsilon^{nop}, \tag{7}$$

and

$$\epsilon_{lrs} = |detM|\frac{\partial x^t}{\partial x'^l}\frac{\partial x^u}{\partial x'^r}\frac{\partial x^v}{\partial x'^s}\epsilon_{tuv}. \tag{8}$$

Then we find that

$$T' = \epsilon'^{klm}\epsilon'_{lrs}A'^rB'^sB'_mA'_k = |detM|^4\delta^u{}_j\delta^q{}_n\delta^v{}_y\delta^w{}_p\delta^t{}_o\epsilon^{nop}\epsilon_{tuv}A^jB^yB_wA_q = |detM|^4\epsilon^{qtw}\epsilon_{tjy}A^jB^yB_wA_q = |detM|^4T. \tag{9}$$

We see that if $|detM| = -1$ then

$$T' = T, \tag{10}$$

indicating that T is a scalar.

d) $T = 0$ if $\mathbf{A}$ and $\mathbf{B}$ are parallel (or antiparallel) to each other because in that case $\cos\theta = \pm 1$ and thus,

$$(\mathbf{A}.\mathbf{B})^2 = A^2 B^2. \tag{11}$$

Problem 2:

a) The field strength tensor is given by

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha. \tag{12}$$

We know that $\partial_\beta = (\partial_0, \vec{\nabla})$ then

$$\partial^\alpha = g^{\alpha\beta}(\partial_0, \vec{\nabla}) = (\partial_0, -\vec{\nabla}). \tag{13}$$

We know that

$$A^\alpha = (\Phi, \mathbf{A}) = (-E_o x^1, -\frac{B_0}{2}x^2, \frac{B_0}{2}x^1, 0). \tag{14}$$

Since A^α is independent of $t = x_0/c$ and x_3 and $A^3 = 0$, and since $F^{\alpha\beta}$ is antisymmetric, we only need to calculate three of its 16 components:

$$F^{01} = -\partial^1 A^0 = \frac{\partial A^0}{\partial x^1} = -E_0, \tag{15}$$

$$F^{02} = -\partial^2 A^0 = \frac{\partial A^0}{\partial x^2} = 0, \tag{16}$$

and

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -\frac{\partial A^2}{\partial x^1} + \frac{\partial A^1}{\partial x^2} = \frac{B_0}{2} + \frac{B_0}{2} = B_0. \tag{17}$$

Then,

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & -B_0 & 0 \\ 0 & B_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

b) Since in general

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{19}$$

we see comparing Eq.(19) with Eq.(18) that $\mathbf{E} = (E_0, 0, 0)$ and $\mathbf{B} = (0, 0, B_0)$

c) Let's write the Lorentz transformation by components:

$$x'^0 = \gamma(x^0 - \beta_1 x^1 - \beta_2 x^2 - \beta_3 x^3), \tag{20}$$

$$x'^i = x^i + \frac{(\gamma - 1)}{\beta^2}(\beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3)\beta_i - \gamma\beta^i x^0, \tag{21}$$

for $i = 1$, 2, and 3. If $\mathbf{v} = (u, u, 0)$ then $\vec{\beta} = (\frac{u}{c}, \frac{u}{c}, 0)$. Then the transformation takes the form:

$$x'^0 = \gamma[x^0 - \frac{u}{c}(x^1 + x^2)], \tag{22}$$

$$x'^1 = x^1 + \frac{(\gamma - 1)}{2}(x^1 + x^2) - \gamma\frac{u}{c}x^0, \tag{23}$$

$$x'^2 = x^2 + \frac{(\gamma - 1)}{2}(x^1 + x^2) - \gamma\frac{u}{c}x^0, \tag{24}$$

$$x'^3 = x^3. \tag{25}$$

Then,

$$M^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} = \begin{pmatrix} \gamma & \frac{-\gamma u}{c} & \frac{-\gamma u}{c} & 0 \\ \frac{-\gamma u}{c} & \frac{(1+\gamma)}{2} & \frac{(\gamma-1)}{2} & 0 \\ \frac{-\gamma u}{c} & \frac{(\gamma-1)}{2} & \frac{(1+\gamma)}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{26}$$

d) We know that

$$A'^\alpha = M^\alpha{}_\beta A^\beta, \tag{27}$$

then

$$A'^0 = -\gamma E_0 x^1 + \frac{\gamma u}{c}\frac{B_0}{2}(x^2 - x^1) = \Phi' \tag{28}$$

$$A'^1 = \frac{\gamma u}{c}E_0 x^1 - \frac{B_0}{2}x^2\frac{(1+\gamma)}{2} + \frac{B_0}{2}x^1\frac{(\gamma - 1)}{2} = A'_x \tag{29}$$

$$A'^2 = \frac{\gamma u}{c}E_0 x^1 - \frac{B_0}{2}x^2\frac{(\gamma - 1)}{2} + \frac{B_0}{2}x^1\frac{(\gamma + 1)}{2} = A'_y \tag{30}$$

$$A'^3 = A^3 = 0 = A'_z. \tag{31}$$

e) Notice that $E'_y = F'^{20}$, thus we need to calculate F'^{20} knowing that $F'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}F^{\alpha\beta}$; since from Eq.(18) we see that the only non-zero components of $F^{\alpha\beta}$ are F^{01} , F^{10} , F^{12} , and F^{21} then

$$F'^{20} = \frac{\partial x'^2}{\partial x^0}\frac{\partial x'^0}{\partial x^1}F^{01} + \frac{\partial x'^2}{\partial x^1}\frac{\partial x'^0}{\partial x^0}F^{10} + \frac{\partial x'^2}{\partial x^1}\frac{\partial x'^0}{\partial x^2}F^{12} + \frac{\partial x'^2}{\partial x^2}\frac{\partial x'^0}{\partial x^1}F^{21}. \tag{32}$$

In terms of E_0, B_0, u, c and γ we obtain:

$$E'_y = \frac{(\gamma - 1)}{2} E_0 + \gamma \frac{u}{c} B_0. \tag{33}$$

Problem 3:

a)

i) We see that all four charges are at the same distance r_q from the origin given by $r_q = \sqrt{2}d$. We also see that $\theta_{q+} = \pi/4$ for the two positive charges while $\theta_{q-} = 3\pi/4$ for the two negative charges. Finally the two charges with positive y coordinate are located at $\phi_{q+} = \pi/2$ while the two charges with negative y coordinate are located at $\phi_{q-} = 3\pi/2$. Then the four locations in spherical coordinates become:

$$(0, d, d) \rightarrow (\sqrt{2}d, \frac{\pi}{4}, \frac{\pi}{2}), \tag{34}$$

$$(0, -d, d) \rightarrow (\sqrt{2}d, \frac{\pi}{4}, \frac{3\pi}{2}), \tag{35}$$

$$(0, d, -d) \rightarrow (\sqrt{2}d, \frac{3\pi}{4}, \frac{\pi}{2}), \tag{36}$$

$$(0, -d, -d) \rightarrow (\sqrt{2}d, \frac{3\pi}{4}, \frac{3\pi}{2}). \tag{37}$$

ii) We use the expansion in terms of spherical harmonics for $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ that we found in class replacing $\mathbf{r}'$ by the location of each of the 4 charges found in (a-i) and using that $\Phi_q = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ for each charge. Using the superposition principle we obtain:

$$\Phi(r,\theta,\phi) = \frac{q}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r_<^l}{r_>^{l+1}}\frac{1}{2l+1}Y_{lm}(\theta,\phi)[Y^*_{lm}(\frac{\pi}{4},\frac{\pi}{2}) + Y^*_{lm}(\frac{\pi}{4},\frac{3\pi}{2}) - Y^*_{lm}(\frac{3\pi}{4},\frac{\pi}{2}) - Y^*_{lm}(\frac{3\pi}{4},\frac{3\pi}{2})], \tag{38}$$

where $r_<$ ($r_>$) is the smaller (larger) between r and $\sqrt{2}d$.

iii) Now we need to look at the explicit form of the $Y^*_{lm}(\theta,\phi)$ in Eq.(38). We know that

$$Y^*_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{-im\phi}. \tag{39}$$

Then using Eq.(39), we find that

$$[Y^*_{lm}(\frac{\pi}{4},\frac{\pi}{2}) + Y^*_{lm}(\frac{\pi}{4},\frac{3\pi}{2}) - Y^*_{lm}(\frac{3\pi}{4},\frac{\pi}{2}) - Y^*_{lm}(\frac{3\pi}{4},\frac{3\pi}{2})] = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}[P_l^m(\frac{\sqrt{2}}{2}) - P_l^m(-\frac{\sqrt{2}}{2})](e^{-im\frac{\pi}{2}} + e^{-im\frac{3\pi}{2}}). \tag{40}$$

We see that the sum of the two exponentials vanishes if m is odd. If m is even, i.e. $m = 2s$ it takes the value 2 if s is even and -2 if s is odd. We also know that the associated Legendre polynomials are even if $l+m$ is even and odd if $l+m$ is odd. The factor in brackets with the sum of two associated Legendre polynomials will then vanish if $l+m$ is even. If $l+m$ is odd it does not vanish. Since m is even then l has to be odd, i.e. $l = 2j+1$. This means that for the non-vanishing terms the bracket can be replaced by $2P_{2j+1}^{2s}(\frac{\sqrt{2}}{2})$. Thus we find that the non-vanishing terms are of the form:

$$[Y_{2j+1,2s}^*(\frac{\pi}{4},\frac{\pi}{2})+Y_{2j+1,2s}^*(\frac{\pi}{4},\frac{3\pi}{2})-Y_{2j+1,2s}^*(\frac{3\pi}{4},\frac{\pi}{2})-Y_{2j+1,2s}^*(\frac{3\pi}{4},\frac{3\pi}{2})] = 4(-1)^s\sqrt{\frac{(4j+2)}{4\pi}\frac{(2j+1-2s)!}{(2j+1+2s)!}}P_{2j+1}^{2s}(\frac{\sqrt{2}}{2}). \tag{41}$$

Thus, only terms with l odd and m even appear in the expansion. If $l=2j+1$ and $m=2s$ since $-l \leq m \leq l$ we find that $-2j \leq 2s \leq 2j$ and thus $-j \leq s \leq j$.

iv)Now we can write the potential in Eq.(38) as

$$\Phi(r,\theta,\phi) = \frac{4q}{\epsilon_0}\sum_{j=0}^{\infty}\sum_{s=-j}^{j}\frac{r_<^{2j+1}}{r_>^{2j+2}}\frac{(-1)^s}{4j+3}\sqrt{\frac{(4j+3)}{4\pi}\frac{(2j+1-2s)!}{(2j+1+2s)!}}P_{2j+1}^{2s}(\frac{\sqrt{2}}{2})Y_{2j+1,2s}(\theta,\phi), \tag{42}$$

where $r_<$ ($r_>$) is the smaller (larger) between r and $\sqrt{2}d$.

b)

i) If $d \to 0$ with $qd = p$ with p a constant we see that all the charges will be located at a point. The total charge of the array is zero but we expect it to have a dipole momentum pointing along the z-axis. Thus, I expect that the potential will depend only on r and θ.

ii) To take the limit we need to consider the potential in Eq.(42) with $r_< = \sqrt{2}d$ and $r_> = r$ then

$$\lim_{d\to 0}\frac{4q}{\epsilon_0}\sum_{j=0}^{\infty}\sum_{s=-j}^{j}\frac{(\sqrt{2}d)^{2j+1}}{r^{2j+2}}\frac{(-1)^s}{4j+3}\sqrt{\frac{(4j+3)}{4\pi}\frac{(2j+1-2s)!}{(2j+1+2s)!}}P_{2j+1}^{2s}(\frac{\sqrt{2}}{2})Y_{2j+1,2s}(\theta,\phi) \tag{43}$$

we see that only the term with $j=0$ will survive since for $j=0$ we obtain $lim_{d\to 0}qd = p$ while for $j>0$ we see that $lim_{d\to 0}qd^{2j+1} = 0$.

iii) Then we obtain that

$$\lim_{d\to 0}\Phi(r,\theta,\phi) = \frac{4p}{\epsilon_0}\frac{(\sqrt{2})}{r^2}\frac{1}{3}\sqrt{\frac{3}{4\pi}}P_1(\frac{\sqrt{2}}{2})Y_{1,0}(\theta,\phi).(44)$$

Since $Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}P_1(\cos\theta)$ and $P_1(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}$ we find that the potential is independent of ϕ and given by

$$\Phi(r,\theta) = \frac{p}{\pi\epsilon_0 r^2}P_1(\cos\theta). \tag{45}$$

c)

i) The potential inside the spehere will depend on r and θ because the problem has azimuthal symmetry.

ii) I propose

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) + \frac{p}{\pi\epsilon_0 r^2}P_1(\cos\theta). \tag{46}$$

I used the principle of superposition to take into account the contribution of the point-like dipole at the origin. Then I only have a set of coefficients A_l for the positive powers of r that I will determine by using the boundary condition that the potential is zero on the sphere surface, i.e., $\Phi(r = a, \theta) = 0$.

iii) Now I use the b.c. and solve for A_l. At $r = a$:

$$\Phi(a, \theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) + \frac{p}{\pi\epsilon_0 a^2} P_1(\cos\theta) = 0. \tag{47}$$

Then, $A_l = 0$ for all $l \neq 1$ and for $l = 1$:

$$A_1 = -\frac{p}{\pi\epsilon_0 a^3}. \tag{48}$$

Then

$$\Phi(r, \theta) = \frac{p}{\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{r}{a^3}\right)\cos\theta. \tag{49}$$