Homework #10

Problem 5:

Since we have to calculate the potential produced by the charged ring in all space we know that the Green function is just given by

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}, \tag{1}$$

since we do not need the extra term needed to adjust the potential to zero on a surface at a finite distance from the origin. We also know that from Green's theorem, the potential is given by:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')dV. \tag{2}$$

In this case

$$\rho(\mathbf{r}) = \frac{q}{2\pi a^2}\delta(r-a)\delta(cos\theta). \tag{3}$$

In order to perform the integral in Eq.(2) easily we will expand Eq.(1) in spherical harmonics. We saw in class that

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r_<^l}{r_>^{l+1}}\frac{Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi')}{(2l+1)}. \tag{4}$$

Plugging (3) and (4) in (2) we obtain:

$$\Phi(\mathbf{r}) = \frac{q4\pi}{4\pi\epsilon_0 2\pi a^2}\int_0^\infty r'^2dr'\int_{-1}^{1}dcos(\theta')\int_0^{2\pi}d\phi'\delta(r'-a)\delta(cos\theta')\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r_<^l}{r_>^{l+1}}\frac{Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi')}{(2l+1)} =$$

$$= \frac{q}{2\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_l^m(\theta,\phi)}{(2l+1)}\frac{r_<^l}{r_>^{l+1}}\int_0^{2\pi}d\phi' Y_l^{m*}(\pi/2,\phi'), \tag{5}$$

but

$$\int_0^{2\pi}d\phi' Y_l^{m*}(\pi/2,\phi') = 2\pi\sqrt{\frac{2l+1}{4\pi}}P_l(0)\delta_{m,0}. \tag{6}$$

Then, replacing (6) in (5) we obtain:

$$\Phi(\mathbf{r}) = \frac{q}{\epsilon_0}\sum_{l=0}^{\infty}\frac{Y_l^0(\theta,\phi)}{(2l+1)}\frac{r_<^l}{r_>^{l+1}}\sqrt{\frac{2l+1}{4\pi}}P_l(0). \tag{7}$$

Using (15.96) we obtain:

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\sum_{j=0}^{\infty}\frac{(-1)^j(2j)!}{2^{2j}(j!)^2}\frac{r_<^{2j}}{r_>^{2j+1}}P_{2j}(cos(\theta)), \tag{8}$$

where $r_<$ ($r_>$) is the smaller (larger) between r and a.