Homework #10

Problem 6:

In class we found that the Green function for a volume inside a sphere of radius R is given by

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R}{r'|\mathbf{r}-\frac{R^2\hat{\mathbf{n}}}{r'}|}. \quad (1)$$

The expansion for the first term in terms of spherical harmonics has been presented in the previous problem and it is given by:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} \frac{Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi')}{(2l+1)}. \quad (2)$$

The expansion of the second term has a similar form but we know that $r < R^2/r'$ because $r < R$ always, and $r' < R$ always which means that $R/r' > 1$ and thus $R^2/r' > R$. Then

$$\frac{1}{|\mathbf{r}-\frac{R^2\hat{\mathbf{n}}}{r'}|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{r^l r'^{l+1}}{(R^2)^{l+1}} \frac{Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi')}{(2l+1)}. \quad (3)$$

Replacing (2) and (3) in (1) we obtain:

$$G(\mathbf{r},\mathbf{r}') = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{(2l+1)}[\frac{r_<^l}{r_>^{l+1}} - \frac{r^l r'^l}{R^{2l+1}}]Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi'). \quad (4)$$

where $r_<$ ($r_>$) is the smaller (larger) between r and r'.