Homework #10

Problem 4:

a) From Table 12.3 we see that:

$$x^2 = r^2 sin^2\theta cos^2\phi = r^2[\frac{\sqrt{4\pi}}{3}Y_0^0 - \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_2^0 + \sqrt{\frac{2\pi}{15}}(Y_2^2 + Y_2^{-2})].$$

$$y^2 = r^2 sin^2\theta sin^2\phi = r^2[\frac{\sqrt{4\pi}}{3}Y_0^0 - \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_2^0 - \sqrt{\frac{2\pi}{15}}(Y_2^2 + Y_2^{-2})].$$

$$z^2 = r^2 cos^2\theta = r^2[\frac{\sqrt{4\pi}}{3}Y_0^0 + \frac{2}{3}\sqrt{\frac{4\pi}{5}}Y_2^0]$$

$$xy = r^2 sin^2\theta cos\phi sin\phi = r^2 i\sqrt{\frac{2\pi}{15}}(Y_2^{-2} - Y_2^2).$$

$$yz = r^2 cos\theta sin\theta sin\phi = r^2 i\sqrt{\frac{2\pi}{15}}(Y_2^{-1} + Y_2^1).$$

$$xz = r^2 cos\theta sin\theta cos\phi = r^2 i\sqrt{\frac{2\pi}{15}}(Y_2^{-1} - Y_2^1).$$

b)

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\mathbf{r})d\tau.$$

The non-diagonal components are identical to the expressions given in part a) multiplied by 3. For the diagonal components we get:

$$x^2 = r^2[-\sqrt{\frac{4\pi}{5}}Y_2^0 + 3\sqrt{\frac{2\pi}{15}}(Y_2^2 + Y_2^{-2})].$$

$$y^2 = r^2[-\sqrt{\frac{4\pi}{5}}Y_2^0 - 3\sqrt{\frac{2\pi}{15}}(Y_2^2 + Y_2^{-2})].$$

$$z^2 = r^2 2\sqrt{\frac{4\pi}{5}}Y_2^0$$

c) With the form given in b) the tensor is traceless since

$$tr(x_i x_j) = x_i x_i = r^2,$$

then

$$tr(Q_{ij}) = 3x_i x_i - 3r^2 = 3r^2 - 3r^2 = 0.$$