Homework #11

Problem 3:

Since the potential on the surfaces is given we need to use the Green function for Dirichlet boundary conditions that was obtained in class:

$$G_D(x,y,x',y') = 8\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\frac{\sinh[(b-y_>)\frac{n\pi}{a}]\sinh(y_<\frac{n\pi}{a})}{\sinh(\frac{bn\pi}{a})}, \tag{1}$$

where $y_>(y_<)$ is the larger (smaller) between y and y'.

In the problem we are considering the density of charge is

$$\rho(x',y') = \sigma_0\theta(y'-b/2), \tag{2}$$

where $\theta(y'-b/2) = 1$ for $y' \leq b/2$ and $y' = 0$ for $y' > b/2$.

The potential inside the volume defined by $0 \leq x \leq a$ and $0 \leq y \leq b$ is given by:

$$\Phi(x,y) = \frac{1}{4\pi\epsilon_0}\int_V G(x,x',y,y')\rho(x',y')dx'dy' - \frac{1}{4\pi}\oint_S \Phi_s\frac{\partial G_D}{\partial n'}dS'. \tag{3}$$

Since $\Phi_s = 0$ on all the surfaces the surface integral does not contribute. Let us then calculate the volume integral. Because of Eq.(2) we obtain:

$$\Phi(x,y) = \frac{\sigma_0}{4\pi\epsilon_0}\int_0^a dx'\int_0^{b/2} dy' G(x,x',y,y'). \tag{4}$$

Let's first calculate the potential $\Phi(x,y)$ for $y > b/2$. In this case $y_< = y'$ and $y_> = y$ in Eq.(1) and we obtain:

$$\Phi(x,y) = 2\frac{\sigma_0}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{a}\frac{\sinh[(b-y)\frac{n\pi}{a}]}{\sinh(\frac{bn\pi}{a})}\int_0^a dx'\sin\frac{n\pi x'}{a}\int_0^{b/2} dy'\sinh(y'\frac{n\pi}{a}). \tag{5}$$

Let's evaluate the two integrals:

$$\int_0^a dx'\sin\frac{n\pi x'}{a} = (\frac{-a}{n\pi})\cos\frac{n\pi x'}{a}|_0^a = (\frac{-a}{n\pi})[(-1)^n - 1]. \tag{6}$$

Eq.(6) equals $(\frac{2a}{n\pi})$ if n is odd and it vanishes if n is even, while for the second integral we have:

$$\int_0^{b/2} dy'\sinh(y'\frac{n\pi}{a}) = (\frac{a}{n\pi})\cosh\frac{n\pi y'}{a}|_0^{b/2} = (\frac{a}{n\pi})(\cosh\frac{bn\pi}{2a} - 1). \tag{7}$$

Thus, for $y \geq b/2$ we obtain:

$$\Phi(x,y) = 4\frac{\sigma_0 a^2}{\pi^3\epsilon_0}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}\sin\frac{(2j+1)\pi x}{a}\frac{\sinh[(b-y)\frac{(2j+1)\pi}{a}]}{\sinh(\frac{b(2j+1)\pi}{a})}(\cosh\frac{(2j+1)b\pi}{2a} - 1). \tag{8}$$

Now let's calculate the potential $\Phi(x,y)$ for $y \leq b/2$. In this case we have to split the integral over y' in two pieces so that for $y' < y$ we will use $y_< = y'$ and $y_> = y$ in Eq.(1) and for $y' > y$ we will use $y_< = y$ and $y_> = y'$ in Eq.(1):

$$\Phi(x,y) = 2\frac{\sigma_0}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\frac{\sin\frac{n\pi x}{a}}{\sinh(\frac{bn\pi}{a})}\int_0^a dx'\sin\frac{n\pi x'}{a}[\sinh[(b-y)\frac{n\pi}{a}]\int_0^y dy'\sinh(y'\frac{n\pi}{a}) + \sinh(y\frac{n\pi}{a})\int_y^{b/2} dy'\sinh[(b-y')\frac{n\pi}{a}]. \tag{9}$$

The integral over x' is the same as in Eq.(6) while for the integrals on y' we obtain:

$$\int_0^y dy' \sinh(y'\frac{n\pi}{a}) = (\frac{a}{n\pi}) \cosh \frac{n\pi y'}{a}|_0^y = (\frac{a}{n\pi})(\cosh \frac{ny\pi}{a} - 1). \tag{10}$$

$$\int_y^{b/2} dy' \sinh[(b-y')\frac{n\pi}{a}] = \int_y^{b/2} dy' [\sinh(\frac{bn\pi}{a}) \cosh(\frac{y'n\pi}{a}) - \cosh(\frac{bn\pi}{a}) \sinh(\frac{y'n\pi}{a})] =$$

$$\frac{a}{n\pi}[\sinh(\frac{bn\pi}{a}) \sinh(\frac{y'n\pi}{a}) + \cosh(\frac{bn\pi}{a}) \cosh(\frac{y'n\pi}{a})]|_y^{b/2} =$$

$$-\frac{a}{n\pi} \cosh(\frac{(b-y')n\pi}{a})|_y^{b/2} = \frac{a}{n\pi}[\cosh(\frac{(b-y)n\pi}{a}) - \cosh(\frac{bn\pi}{2a})]. \tag{11}$$

Thus, for $y \leq b/2$ we obtain:

$$\Phi(x,y) = 4\frac{\sigma_0 a^2}{\pi^3 \epsilon_0} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^3} \frac{\sin \frac{(2j+1)\pi x}{a}}{\sinh(\frac{b(2j+1)\pi}{a})} \{\sinh[(b-y)\frac{(2j+1)\pi}{a}](\cosh \frac{ny\pi}{a} - 1) +$$

$$\sinh[y\frac{(2j+1)\pi}{a}][\cosh(\frac{(b-y)n\pi}{a}) - \cosh(\frac{bn\pi}{2a})]\}.$$

(12)