Homework #2

Problem 1:

a) From the figure we see that the coordinates of $\mathbf{r}$ in S are given by:

$$x = r\cos\theta \tag{1}$$

$$y = r\sin\theta \tag{2}$$

b) Let us find an expression for the axis $\mathbf{e}'_i$ in terms of the cartesian basis vectors $\mathbf{e}_j$. From the figure:

$$\mathbf{e}'_1 = \cos\gamma\mathbf{e}_1 - \sin\gamma\mathbf{e}_2 \tag{3}$$

$$\mathbf{e}'_2 = -\sin\beta\mathbf{e}_1 + \cos\beta\mathbf{e}_2 \tag{4}$$

We see that

$$(\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\gamma & -\sin\beta \\ -\sin\gamma & \cos\beta \end{pmatrix}. \tag{5}$$

Here

$$A = \begin{pmatrix} \cos\gamma & -\sin\beta \\ -\sin\gamma & \cos\beta \end{pmatrix}. \tag{6}$$

Then

$$\mathbf{e}'_i = \mathbf{e}_j A^j{}_i = A^j{}_i\mathbf{e}_j \tag{7}$$

c) From the figure we see that the covariant components are given by:

$$x'_1 = r\cos(\theta+\gamma) = r\cos\theta\cos\gamma - r\sin\theta\sin\gamma = x_1\cos\gamma - x_2\sin\gamma. \tag{8}$$

$$x'_2 = r\cos(\pi/2+\beta-\theta) = -r\sin(\beta-\theta) = -r\cos\theta\sin\beta + r\sin\theta\cos\beta = -x_1\sin\beta + x_2\cos\beta. \tag{9}$$

We see that x'_i transforms like the axis, i.e.:

$$x'_i = x_j A^j{}_i = A^j{}_i x_j \tag{10}$$

d) From the figure we see that

$$\mathbf{x}'^1 + \mathbf{x}'^2 = \mathbf{r}. \tag{11}$$

Then

$$x_1 = (\mathbf{x}'^1 + \mathbf{x}'^2)\mathbf{e}_1 = x'^1\cos\gamma - x'^2\sin\beta, \tag{12}$$

and

$$x_2 = (\mathbf{x}'^1 + \mathbf{x}'^2)\mathbf{e}_2 = -x'^1 \sin\gamma + x'^2 \cos\beta. \tag{13}$$

The above can be expressed matricially as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\sin\beta \\ -\sin\gamma & \cos\beta \end{pmatrix} \begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix}. \tag{14}$$

Then

$$x^i = A^i{}_j x'^j, \tag{15}$$

where I used that $x_i = x^i$. Clearly to obtain the contravariant variable in terms of the cartesian ones I need to invert the relationship given in Eq. 15 and I get

$$x'^i = M^i{}_j x^j, \tag{16}$$

with

$$M = \begin{pmatrix} \frac{\cos\beta}{\cos(\gamma+\beta)} & \frac{\sin\beta}{\cos(\gamma+\beta)} \\ \frac{\sin\gamma}{\cos(\gamma+\beta)} & \frac{\cos\gamma}{\cos(\gamma+\beta)} \end{pmatrix} \tag{17}$$

Clearly

$$A^i{}_k M^k{}_j = \delta^i{}_j. \tag{18}$$

The contravariant components are obtained by writing explicitely Eq. 16 using Eq. 17:

$$x'^1 = x_1 \frac{\cos\beta}{\cos(\gamma+\beta)} + x_2 \frac{\sin\beta}{\cos(\gamma+\beta)} \tag{19}$$

$$x'^2 = x_1 \frac{\sin\gamma}{\cos(\gamma+\beta)} + x_2 \frac{\cos\gamma}{\cos(\gamma+\beta)} \tag{20}$$

We see from Eq. 16 that x'^i transforms contary to the axis, i.e.they are contravariant components.

e) From Eq. 15 we clearly see that

$$A^i{}_j = \frac{\partial x^i}{\partial x'^j}, \tag{21}$$

and from Eq. 16 we obtain that

$$M^i{}_j = \frac{\partial x'^i}{\partial x^j}. \tag{22}$$

The expressions above can be explicitely verified by taking the corresponding derivatives of Eqs. 12 and Eq. 13 and comparing with Eq. 6 and by taking the corresponding derivatives of Eq. 19 and Eq. 20 and comparing with Eq. 17.

f) In class we wrote the elements of the matrix A in terms of the angle α formed by the axis x'^1 and $x'2$. This means that in order to make comparisons we need to express β and γ in terms of α. Looking at the figure we notice that $\gamma = 0$ and $\beta = \frac{3\pi}{2} + \alpha$. Replacing these values for γ and β in Eq. 6 we obtain:

$$A = \begin{pmatrix} \cos\gamma & -\sin\beta \\ -\sin\gamma & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos(0) & -\sin(\frac{3\pi}{2}+\alpha) \\ -\sin(0) & \cos(\frac{3\pi}{2}+\alpha) \end{pmatrix} = \begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix}, \tag{23}$$

which is in agreement with the matrix A obtained in class.