Homework #2

Problem 5:

a) In the oblique coordinate system K' defined in class the position vector $\mathbf{r}'$ can be written as

$$\mathbf{r}' = a\hat{\mathbf{e}'}_1 + b\hat{\mathbf{e}'}_2.$$

Are a and b the covariant (perpendicular) or contravariant (parallel) components of $\mathbf{r}'$? Why? Give an explanation based on vectors' properties and another based on tensors' properties.

It is clear that

$$a = x'^1$$

and

$$b = x'^2,$$

i.e., the contravariant (parallel) components of $\mathbf{r}'$ in K' because:
i) as vectors $\mathbf{r}' = x'^1\hat{\mathbf{e}'}_1 + x'^2\hat{\mathbf{e}'}_2$,
clearly seen in the figure with the components that was shown in class (see lectures) and

ii) because as a tensor

$$\mathbf{r}' = x'^i\hat{\mathbf{e}'}_i$$

since the base vectors are covariant the components of r'^i have to be contravariant.
b)

Show that

$$a = \frac{(\mathbf{r}' \times \hat{\mathbf{e}'}_2).(\hat{\mathbf{e}'}_1 \times \hat{\mathbf{e}'}_2)}{|\hat{\mathbf{e}'}_1 \times \hat{\mathbf{e}'}_2|^2},$$

and

$$b = \frac{(\mathbf{r}' \times \hat{\mathbf{e}'}_1).(\hat{\mathbf{e}'}_2 \times \hat{\mathbf{e}'}_1)}{|\hat{\mathbf{e}'}_2 \times \hat{\mathbf{e}'}_1|^2}.$$

In class (see lecture) we showed that

$$x'^1 = x_1 - x_2\text{cotan}\alpha,$$

and

$$x'^2 = x_2\text{cosec}\alpha.$$

Now let's look at the expression given for a. We see that

$$\mathbf{r}' \times \hat{\mathbf{e}'}_2 = r\sin(\alpha - \theta) = r\sin\alpha\cos\theta - r\cos\alpha\sin\theta = x_1\sin\alpha - x_2\cos\alpha, \quad (1)$$

and it points out of the plane.

$$\hat{\mathbf{e}'}_1 \times \hat{\mathbf{e}'}_2 = \sin\alpha, \tag{2}$$

and it points out of the plane, then replacing (1) and (2) in the expression for a we obtain:

$$a = \frac{(x_1 \sin\alpha - x_2 \cos\alpha)\sin\alpha}{\sin^2\alpha} = x_1 - x_2 \text{cotan}\alpha = x'^1.$$

Now let's look at the expression given for b. We see that

$$\mathbf{r}' \times \hat{\mathbf{e}'}_1 = r\sin\theta = x_2, \tag{3}$$

and it points into the plane.

$$\hat{\mathbf{e}'}_2 \times \hat{\mathbf{e}'}_1 = \sin\alpha, \tag{4}$$

and it points into the plane, then replacing (3) and (4) in the expression for b we obtain:

$$b = \frac{x_2 \sin\alpha}{\sin^2\alpha} = x_2 \text{cosec}\alpha = x'^2.$$