Homework #3

Problem 1:

a) We know that

$$B'_i = \frac{\partial \Phi'}{\partial x'^i}. \tag{1}$$

Let's transform B'_i to the orthogonal system x^1, x^2:

$$B'_i = \frac{\partial \Phi'}{\partial x'^i} = \frac{\partial \Phi}{\partial x^j}\frac{\partial x^j}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}B_j. \tag{2}$$

To obtain Eq. 2 we have used that as an scalar $\Phi'(x'^i) = \Phi(x^i)$ and that in the unprimed system $B_j = \frac{\partial \Phi}{\partial x^j}$. The equation shows that

$$B'_i = \frac{\partial x^j}{\partial x'^i}B_j, \tag{3}$$

which means that B_j transforms in a covariant way.

b) We know that

$$B'^i = \frac{\partial \Phi'}{\partial x'_i}. \tag{4}$$

The equations in the solution to Problem 1 of Hw#2 allow us to obtain expressions for the covariant and the contravariant components in the primed system in terms of the cartesian components in the unprimed system. In particular Eq.(10) indicated that

$$x'_i = A^j{}_i x_j. \tag{5}$$

Then, inverting the relationship we obtain:

$$x_j = M^i{}_j x'_i. \tag{6}$$

From Eq. 6 we see that

$$M^i{}_j = \frac{\partial x_j}{\partial x'_i}, \tag{7}$$

but Eq.(16) in the solution for Problem 1 of Hw#2 indicated that

$$x'^i = M^i{}_j x^j, \tag{8}$$

then from Eq. 8 we see that

$$M^i{}_j = \frac{\partial x'^i}{\partial x^j}, \tag{9}$$

then comparing Eq. 7 with Eq. 9 we obtain that:

$$\frac{\partial x_j}{\partial x'_i} = \frac{\partial x'^i}{\partial x^j}. \tag{10}$$

Now let's transform B'^i to the orthogonal system x^1, x^2:

$$B'^i = \frac{\partial \Phi'}{\partial x'_i} = \frac{\partial \Phi}{\partial x^j}\frac{\partial x^j}{\partial x'_i} = \frac{\partial x'^i}{\partial x^j} B^j. \tag{11}$$

To obtain Eq. 11 we have used that as an scalar $\Phi'(x'^i) = \Phi(x^i)$, Eq. 10, and that in the unprimed system $B^j = \frac{\partial \Phi}{\partial x_j}$ (remember that $x_j = x^j$ in a cartesian system). The equation shows that

$$B'^i = \frac{\partial x'^i}{\partial x^j} B^j, \tag{12}$$

which means that B^j transforms in a contravariant way.