

Homework #4

Problem 5 - 4.1.5:

We know that

$$R_{iklm} = -R_{ikml} = -R_{kilm}, \quad (1)$$

where the indices can take the values 0, 1, 2, and 3. The rank 4 tensor has $4^4 = 256$ elements. But due to (1) not all the components are independent. Let's calculate the number of independent components. There are 6 combinations of pairs ik that are independent since i can take 4 values and once it is fixed k cannot be equal to i and only can take 3 values. But since the elements with indices (ik) and (ki) are related to each other, we are over counting by a factor of 2 the number of possible values of the indices. Thus, the actual number of independent combinations of the indices is $4 \times 3/2 = 6$. Exactly the same occurs for the pairs lm so the total number of independent elements is $6 \times 6 = 36$.

Now they tell us that

$$R_{iklm} = R_{lmik}. \quad (2)$$

Of the 36 components that are independent we need to see how many elements are different under condition (2). Notice that the first pair can take 6 values namely: (01), (02), (03), (12), (13), and (23) for the first of these cases the second pair can take 6 values, for the second 5 values since R_{0201} will not be independent because it is equal to R_{0102} which has already been counted; for (03) there are 4 possible values of the other pair, and 3, 2 or 1 for the last 3 values of the first pair of indices. For the last pair (23) we see that R_{2323} is the only independent component when we assign (23) to the first pair. Thus, the total number of independent elements is $6 + 5 + 4 + 3 + 2 + 1 = 21$.

If

$$R_{iklm} + R_{ilmk} + R_{imkl} = 0, \quad (3)$$

we see that if $iklm$ has repeated indices (3) does not add anything new because one of the terms will vanish and we are going to obtain one of the previous relationships between two elements. There is only one independent relationship with all the indices different (take for example (0123), since any other ordering of the different indices will give us the same constraint) which means that we are left with $21 - 1 = 20$ independent components.